SAT and Davis-Putnam procedure

A propositional formula ϕ is satisfiable iff there exists a substitution of truth values for its variables that makes it true.

The SAT problem is to, given a formula ϕ , either find such a satisfying assignment or prove that none exists and thus that ϕ is unsatisfiable.

Davis-Putnam procedure II

Preparation: Transform the formula into CNF.

- Delete all clauses that are tautologies.
- Select a variable (propositional symbol).
  - Add all resolvents over that variable.
  - Then delete all clauses with that variable.

Iterate until there are no variables left.

If you end up with NO CLAUSES left then the original set is satisfiable.

If you end up with EMPTY CLAUSE then the original set is not satisfiable.

Example 1

S: { {P, Q} {P, ¬ Q} {¬ P, Q} {¬ P, ¬ Q} }

Adding all resolvents on P and

eliminating P gives { {Q}, {¬ Q}, {Q, ¬ Q} }.

Removing the tautology gives { {Q}, {¬ Q} } .

Adding all resolvents on Q and eliminating Q gives { □ } , i.e

EMPTY CLAUSE

So the output is EMPTY CLAUSE and S is not satisfiable.

Example 2

S: { {P, ¬ Q} {Q} }.

Eliminating P gives { {Q} }.

Eliminating Q gives { } i.e. NO CLAUSE.

Thus the output is NO CLAUSES and S is satisfiable.

Davis-Putnam-Logemann-Loveland Procedure (DPLL)

Rules

UNSAT If F contains $\square$, then F is unsatisfiable.

SAT If F is empty set $\{\}$, then F is satisfiable.

MULT If a literal occurs more than once in a clause, then all but one can be deleted.

SUBS A clause in F can be deleted, if it is a superset of another clause in F.

UNIT An element $\neg$ L of a clause in F can be deleted, if F contains $\{L\}$.

TAUT A clause can be deleted, if it contains a literal and its complement.

PURE A clause can be deleted, if it contains L and ¬ L does not occur in F.

SPLIT

If F is semantically equivalent to a formula of the form

$\{ \{C_1 \vee L\}, ..., \{C_k \vee L\}, \{C_{k+1} \vee \neg L\}, ..., \{C_m \vee \neg L\}, C_{m+1}, ..., C_n\}$

where neither L nor $\neg L$ occur in C_i, 1 $\leq$ i $\leq$ n,

then replace F by the CNF of

$\{C_1, ..., C_k, C_{m+1}, ..., C_n\} \vee \{C_{k+1}, ..., C_m, C_{m+1}, ..., C_n\}$.

DPLL: Examples of rules

UNSAT $\{ \square, C_1, ..., C_n \} \equiv \{\square\}$

MULT $\{L, L, L1, \ldots, Lm\} \equiv \{L, L1, \ldots, Lm\}$.

SUBS $\{ \{L1, \ldots, Lm\}, \{L1, \ldots, Lm, \ldots Lk\}, C1, \ldots, Cn \}$
$\equiv \{ \{L1, \ldots, Lm\}, C1, \ldots, Cn\}$.

UNIT $\{\{L1, \ldots, Lm, L\}, \{\neg L \}\} \equiv \{L1, \ldots, Lm\}, \{\neg L \} \}$

TAUT $\{ \{ L1, \ldots, Lm, L, \neg L\}, C1, \ldots, Cm \} \equiv \{C1, \ldots, Cm \}$.

PURE $\{\{p, \neg q\}, \{\neg r\}\} \not\equiv \{\{p, \neg q\}\}$,
but $\{C1, \ldots, Cm, \{L, L1, \ldots, Ln\}\}$ is unsatisfiable
iff $\{C1, \ldots, Cm\}$ is unsatisfiable, where $\neg$ L
does neither occur in Ci, $1 \leq i \leq$ nor in Lj, $1 \leq j \leq n$.

SPLIT $\{\{p, r\}, \{\neg r\}, \{q\}\} \not\equiv \{\{p\}, \{q\}\} \vee \{\square, \{q\}\}$,
but the rule preserves unsatisfiability.

DPLL: Some properties of the rules

- Let F be a formula in CNF and $F^{'}$ be obtained from F by applying a rule.
- MULT, SUBS, UNIT and TAUT are equivalence preserving, i.e., $F^{'} \equiv F$.
- They are polynomial simplification rules.
- PURE and SPLIT are unsatisfiability preserving,i.e., F is unsatisfiable iff $F^{'}$ is unsatisfiable.

The DPLL-Algorithm

1. Input F. Transform F into CNF.
2. Apply the rules MULT, SUBS, UNIT, TAUT, PURE and SPLIT until SAT or UNSAT become applicable.
3. If SAT is applicable then terminate with **F is satisfiable**.
4. If UNSAT is applicable then terminate with **F is unsatisfiable**.

Remarks

- The algorithm always terminates.
- Whenever the application of a rule in step 3 yields a formula H, then H is unsatisfiable iff F is unsatisfiable.
- The algorithm is sound and complete.

DPLL: An Example

$\{ \{p1, p2\}, \{p4, \neg p2, \neg p3\}, \{ \neg p1, p3\}, \{\neg p4\} \}$

Initialization

$\{ \{p1, p2\}, \{\neg p2, \neg p3\}, \{ \neg p1, p3\}, \{\neg p4 \}\}$

UNIT wrt $\{\neg p4\}$

$\{ \{p1, p2\}, \{\neg p2, \neg p3\}, \{ \neg p1, p3\} \}$

PURE wrt $\neg p4$

$\{ \{p2\}, \{\neg p2, \neg p3\}\} \vee \{\{p3\}, \{\neg p2, \neg p3 \} \}$

SPLIT wrt p1.

$\{ \{p2\}, \{ \neg p3\}\} \vee \{\{p3\}, \{\neg p2 \} \}$

UNIT wrt $\neg$p2 in 1st disjunct and UNIT wrt $\neg$p3 in 2nd one.

$\{ \{ \neg p3\}\} \vee \{ \{\neg p2 \} \}$

PURE wrt p2 in 1st disjunct and PURE wrt p3 in 2nd one .

$\{ \} \vee \{ \}$

PURE wrt $\{ \neg$ p3$\}$ in 1st dis. and PURE wrt $\{\neg$p2 $\}$ in 2nd.

SAT (satisfiable for both branches)