Parameterized Complexity of CSPs

summary

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Constraint Satisfaction Problems

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- Variables $x_1, x_2, \ldots, x_n$, domain D, constraints.
- Each constraint has *arity* a, is applied to a tuple $(i_1, \ldots, i_a) \in [n]^a$, and has relation $R \subseteq D^a$.
- An assignment $\alpha : [n] \to D$ satisfies a constraint if $(\alpha(i_j))_{j=1}^a \in R$.
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- MAX SAT
  - Reasonable parameters: all of the above, number of satisfied constraints, number of unsatisfied constraints, above/below guarantee parameterizations.

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- Negative results:
  - ETH $\implies$ no $|D|^{o(k/\log k)}$ algorithm if the constraint graph is a cubic expander.
  - ETH $\implies$ no $|D|^{o(\mathrm{tw}/\log \mathrm{tw})}$ algorithm.
  - Fundament of many fine-grained parameterized complexity results.

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- For bounded arity of constraints: reduces to the previous slide.
- Large arity examples: CNF SAT, Not-All-Equal SAT.
  - Simple, rather uninteresting reduction to the previous slide.
  - Rather exotic regime of parameters.

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Discussed example: r-CNF SAT, CNF-SAT.

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- (Crowston et al. 2012) If clauses have different length, parameterize above first-point lower bound: para-NP-hard.
- (Gutin et al. 2010) More similar results for constraints being linear equations, for permutation CSPs, etc.

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  - $D = \{0, 1\}, \Gamma = \{1 \to x, x \to 0, x \to y\}$ is DIRECTED MINIMUM CUT.
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- ALMOST 2-SAT is FPT via a reduction to VERTEX COVER ABOVE LP.
  - and VERTEX COVER ABOVE LP is FPT thanks to very strong structural properties of the optimal LP solutions given by Nemhauser-Trotter.

Min UnSAT and flow-augmentation

- Flow-augmentation completes dichotomy for boolean languages.
  - ℓ-CHAIN SAT is FPT ($x \to y \to p \to q$).
  - COUPLED CUT is FPT ($(x \to y) \wedge (p \to q) \wedge (\neg x \vee \neg p)$).
  - BUNDLED CUT with bundles of size 2 is W[1]-hard ($(x \to y) \wedge (p \to q)$).
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- Potential good target for next dichotomy: Temporal CSPs.

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