Spanning spheres in Dirac hypergraphs

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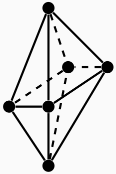

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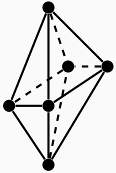

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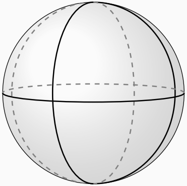

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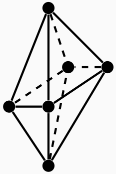

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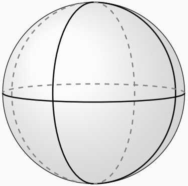

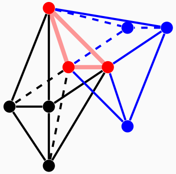

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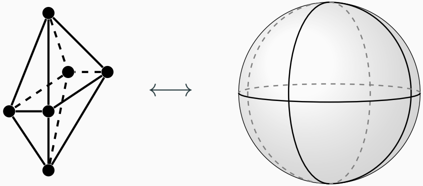

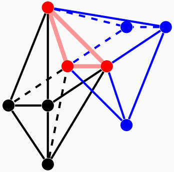

Topological Dirac's Theorem

$\delta(G) > n/2 \implies G$ contains a spanning 1-sphere.

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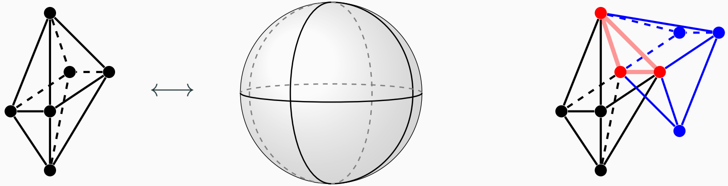

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Question (Conlon and Gowers)

Which degree condition forces a 3-graph to contain a spanning 2-sphere?

$$\delta_2(G) = \min_{\substack{S \subseteq V(G): \\ |S|=2}} \#\{e \in E(G) : e \supseteq S\}$$

2-spheres in 3-graphs - 1/2

A tight walk in a 3-graph is a sequence of vertices such that each set of 3 consecutive vertices induces an edge.

A tight component is a maximal subset of edges such that for any pair of edges e and f, there is a walk from e to f.

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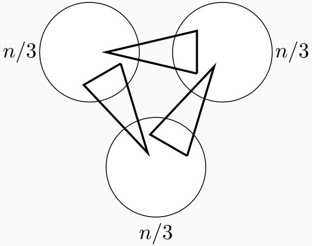
The edges of a (spanning) 2-sphere must all belong to the same (spanning) tight component.

- Edges not belonging to a spanning component are not of any use;
- $\delta_2(G) \geq n/3$ is needed: the graph does not have a spanning tight component.



Theorem (Georgakopoulos, Haslegrave, Montgomery and Narayanan, 2022)

Let G be a 3-graph with $\delta_2(G) \geq (1/3 + o(1))n$
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- Still open for larger uniformity;

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Is the existence of a spanning component the main obstacle?

Can we relax the condition on $\delta_2(G)$ if we assume that G has a unique tight component and this component is spanning?

Our result

Define the supported minimum degree as

$$\delta_2^\star(G) = \min_{\substack{S : |S|=2 \text{ and} \\ S \text{ is contained} \\ \text{in at least one edge}}} \#\{e \in E(G) : e \supseteq S\}\,.$$

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- Result holds for any uniformity, again with $\delta_{k-1}^\star(G) \geq (1/2 + o(1))n$;
- Our proof does not use neither the Absorption Method nor the Regularity Lemma.

Sketch - 1/3

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Cover $V(G)$ with a family of graphs $R_1^\star, \ldots, R_\ell^\star \subseteq G$ such that

- $R_i^\star$ is a *nearly-regular* large blow-up of some small graph R_i, where R_i is tightly connected and inherits the degree condition from G;
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Part 3.

Glue all the spheres along the common facets.

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If $R^\star$ is a blow-up of R, let $\phi : V(R^\star) \to V(R)$ be the projection map.

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- $1/m \ll 1/s$;
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- $R^\star$ blow-up of R, with each part of size roughly m;
- $f_1, f_2 \in E(R^\star)$ such that $\phi(f_1)$ and $\phi(f_2)$ are disjoint.

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3. Glue each sphere S_e with S along g_e.

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d) **Larger uniformity**

Conjecture (Georgakopoulos, Haslegrave, Montgomery and Narayanan, 2022)

Let G be a k-graph with $\delta_{k-1}(G) \geq n/k \implies G$ has a spanning $(k-1)$-sphere.

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Thank you!