Building spanning graphs in the semirandom tree process

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(Joint work with Michael Anastos, Maurício Collares, Joshua Erde, Mihyun Kang, and Gregory Sorkin)

TU Graz

DIMEA Combinatorial Potluck 2024
November 16, 2024

Overview

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  - Proof outline

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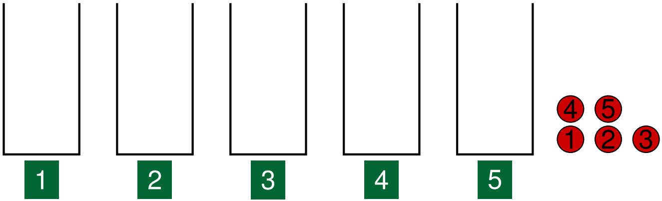

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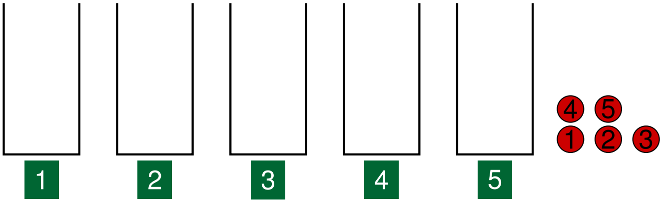

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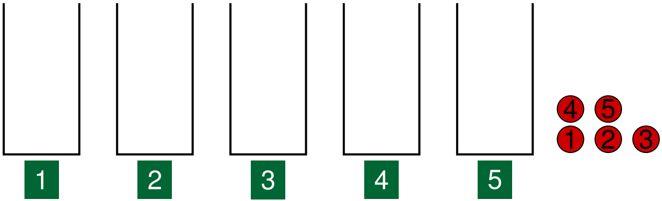

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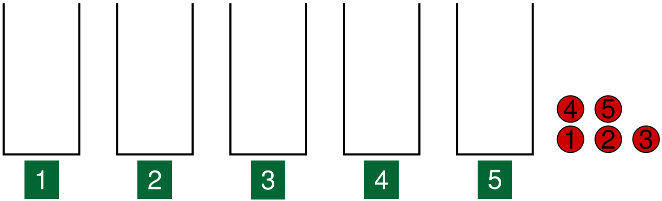

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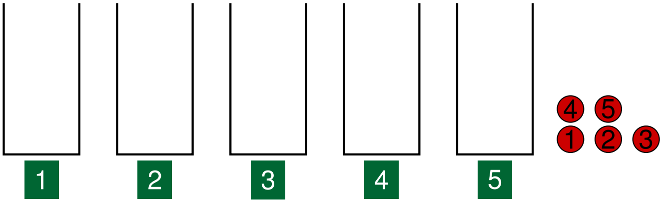

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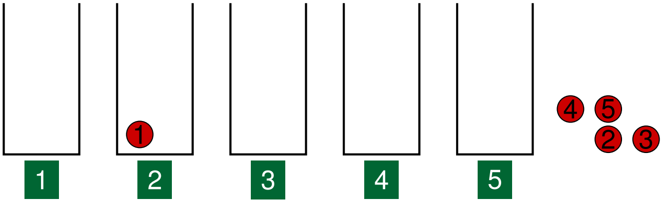

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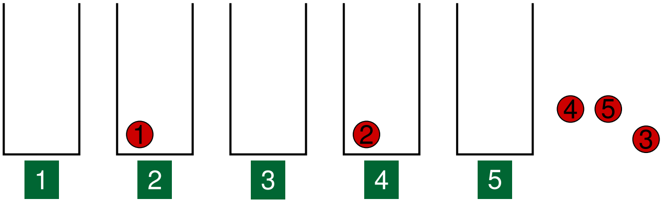

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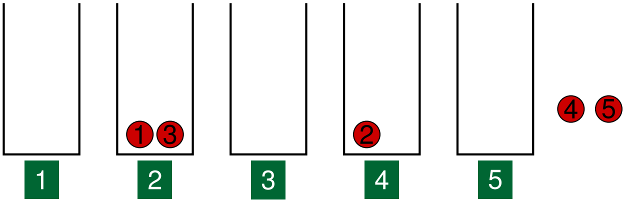

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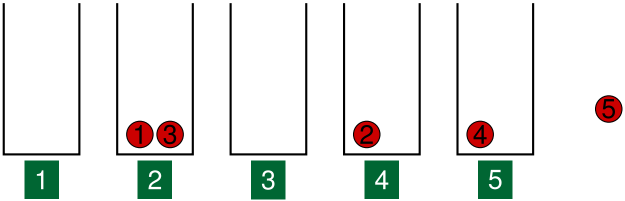

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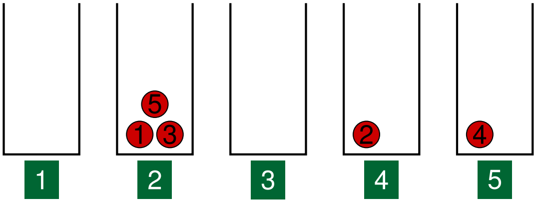

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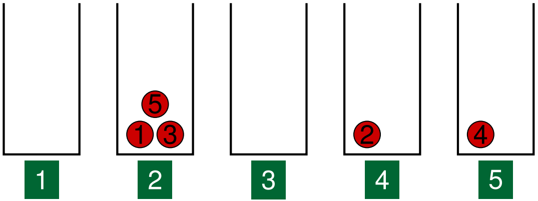

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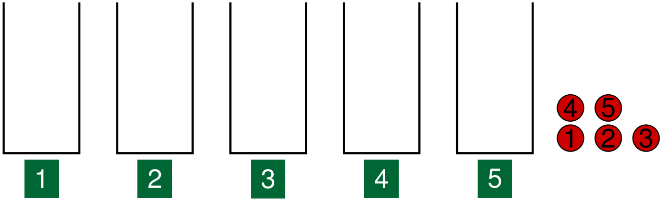

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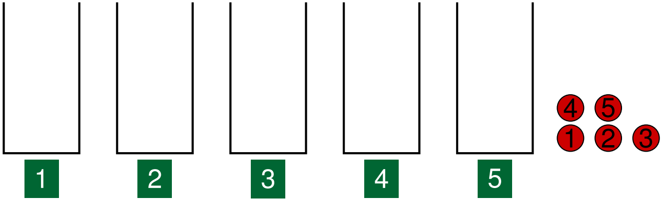

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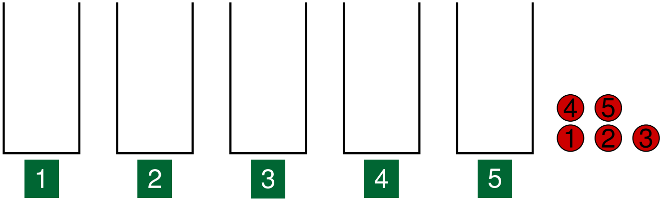

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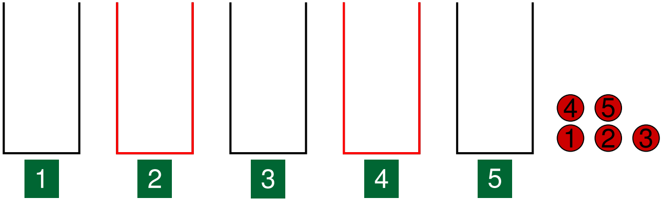

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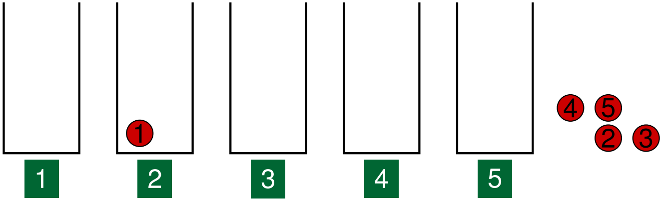

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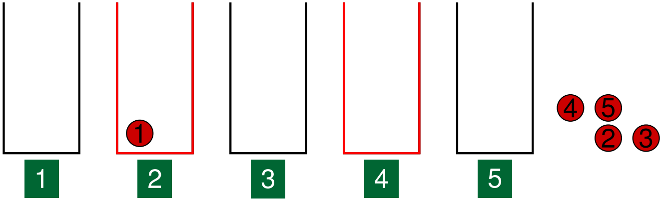

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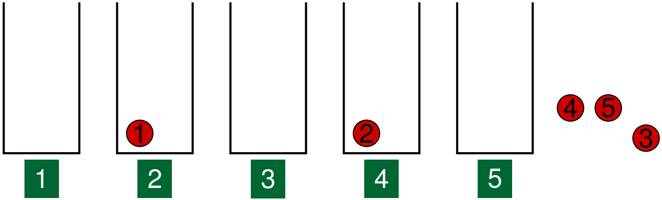

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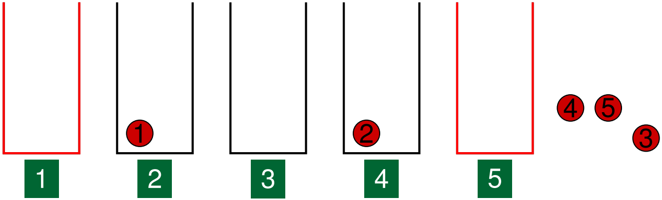

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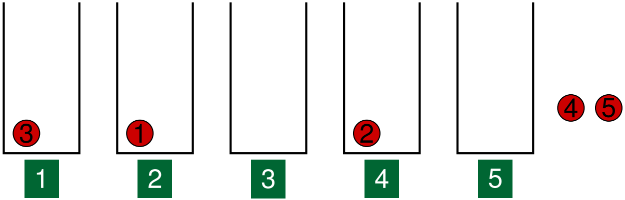

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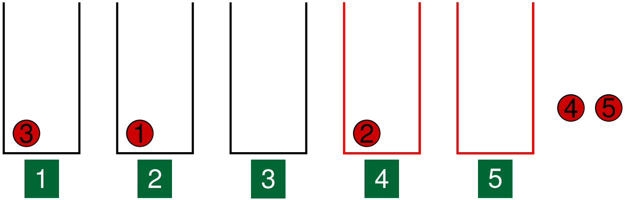

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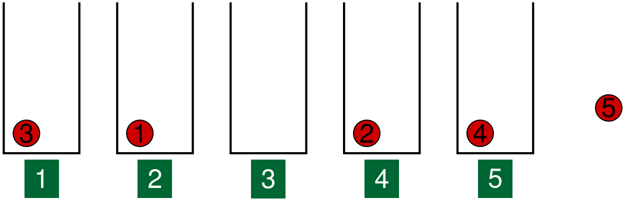

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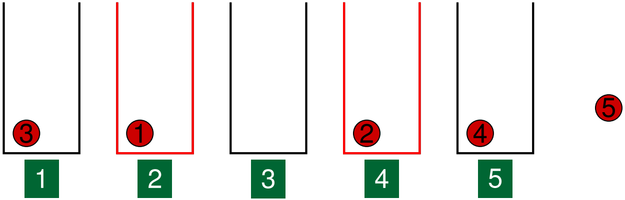

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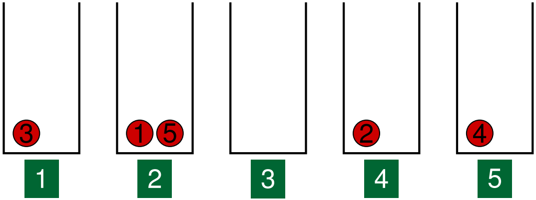

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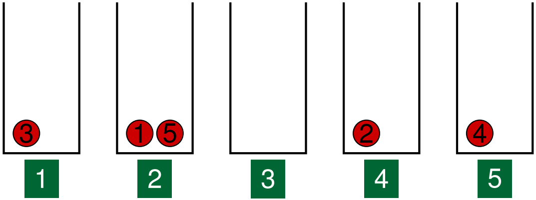

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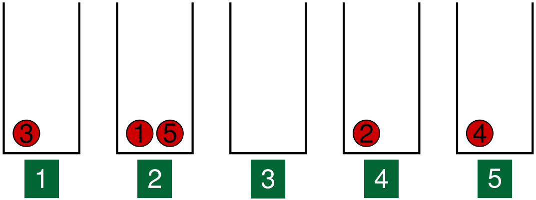

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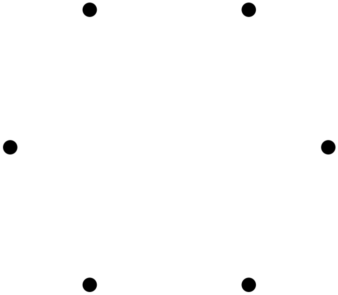

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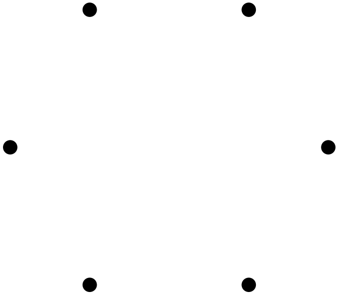

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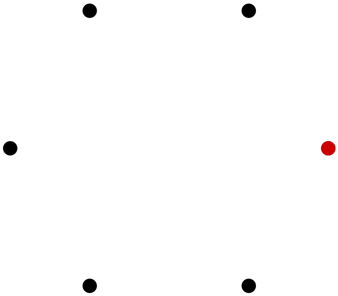

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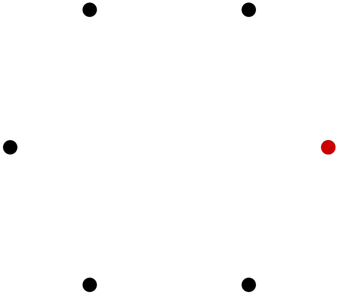

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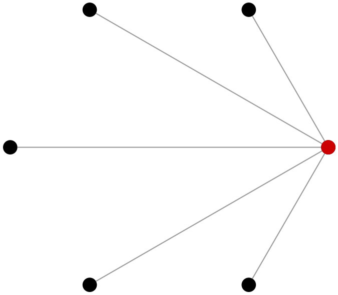

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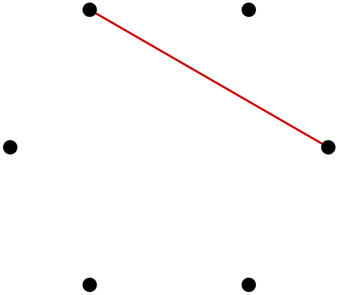

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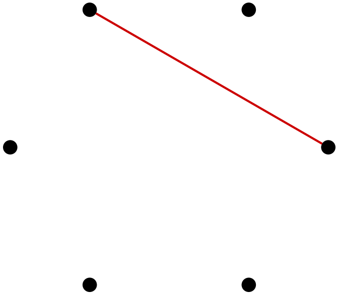

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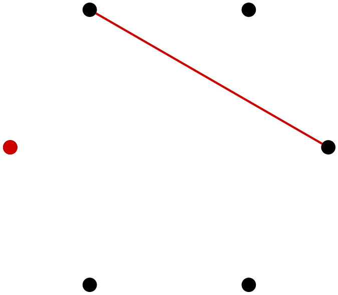

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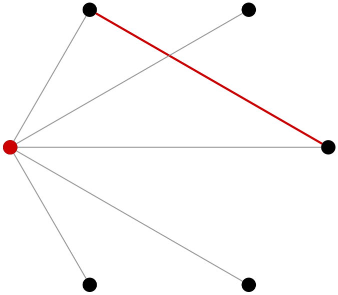

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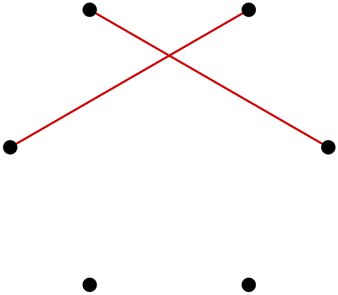

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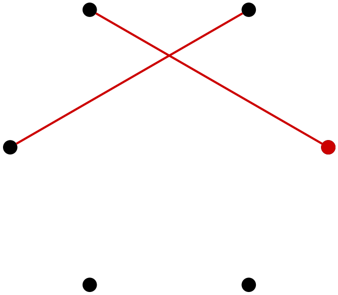

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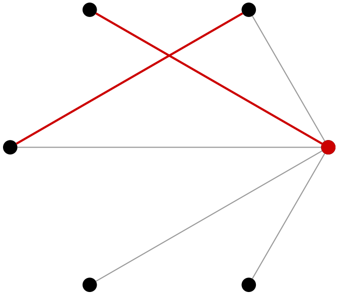

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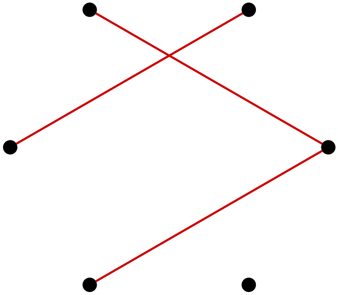

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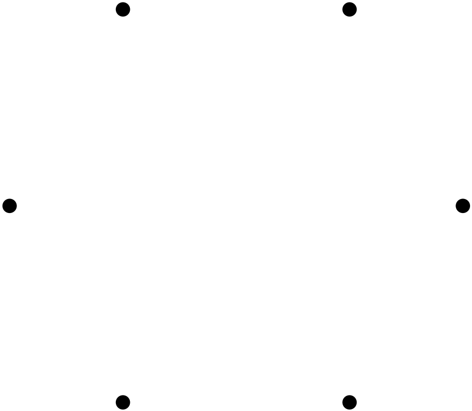
Connectedness in semirandom tree process



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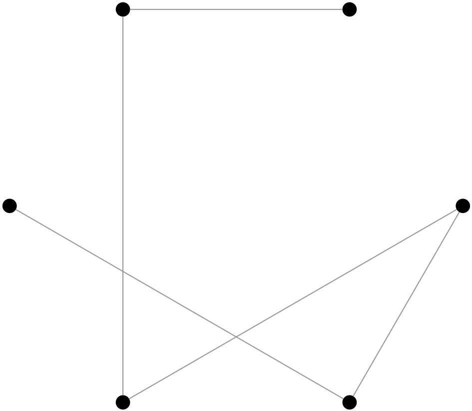

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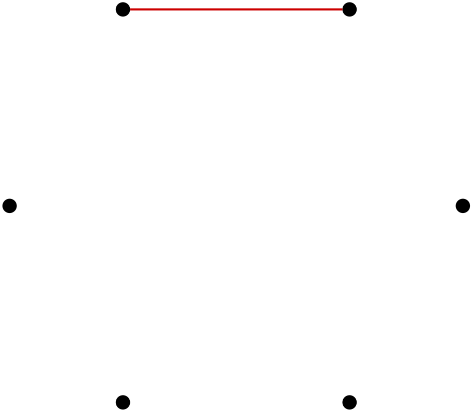

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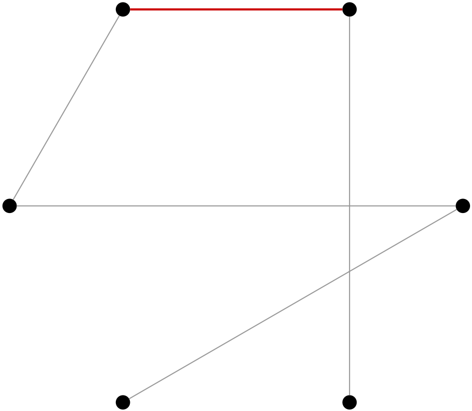

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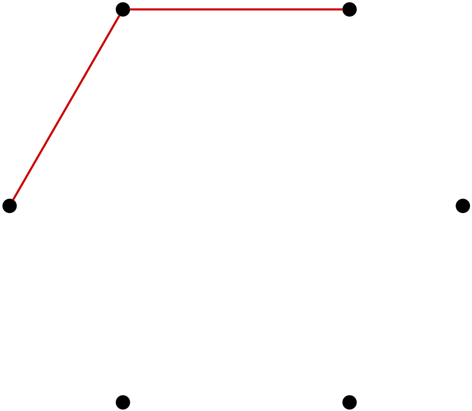

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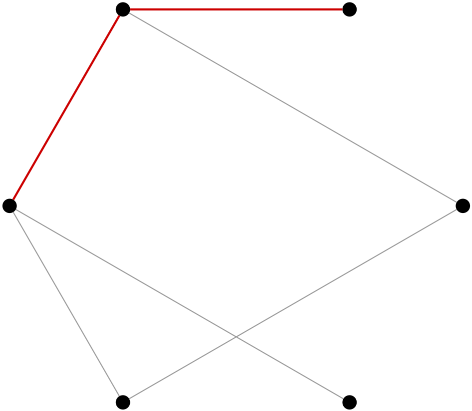

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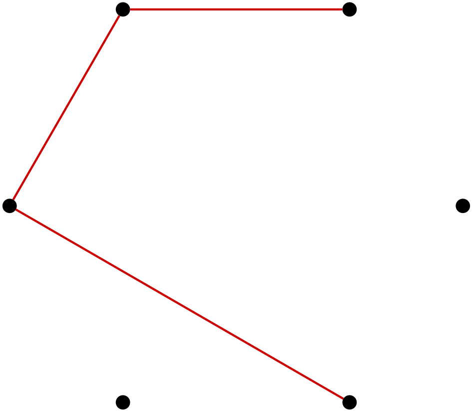

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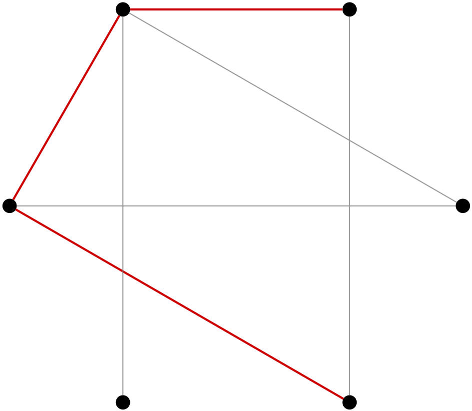

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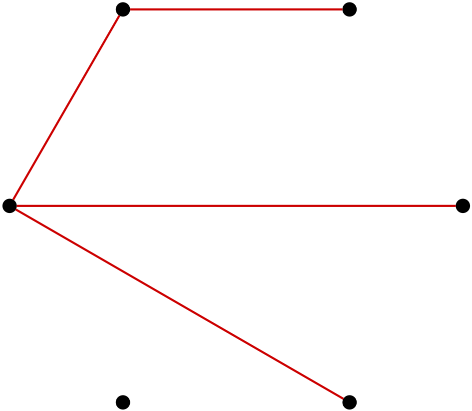

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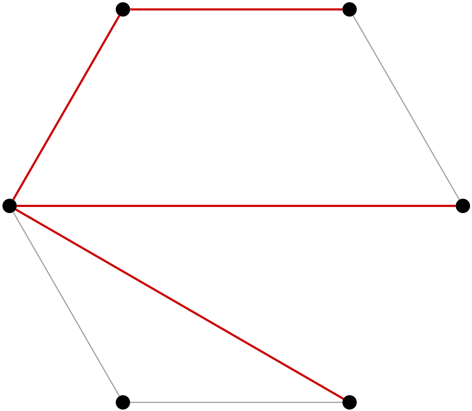

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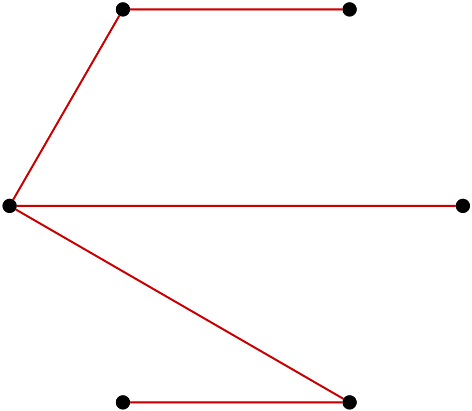

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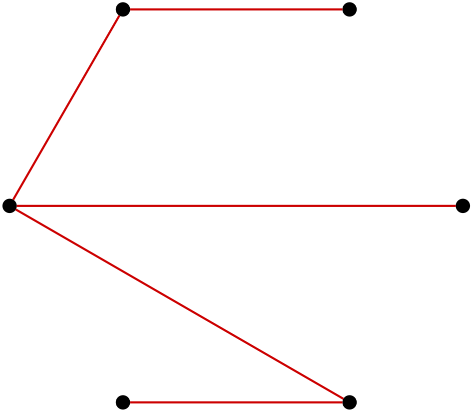

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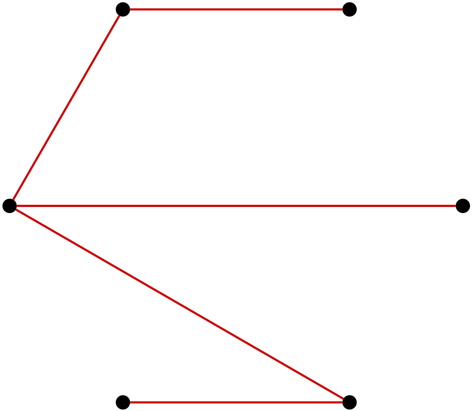

- There exists σ, such that $\tau(\mathcal{P}_C, \sigma) = n - 1$
- Connectedness can be achieved deterministically in optimal time

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Main result

Theorem

[Anastos, Collares, Erde, Kang, S., Sorkin, 2024^{+}]

*Let H be graph with maximum degree Δ. Then, in the semirandom **tree** process, there is a strategy σ such that whp*

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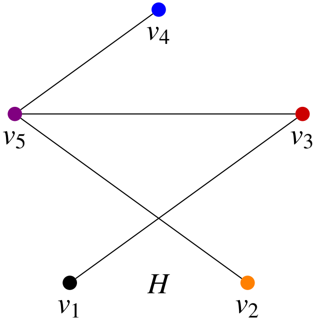

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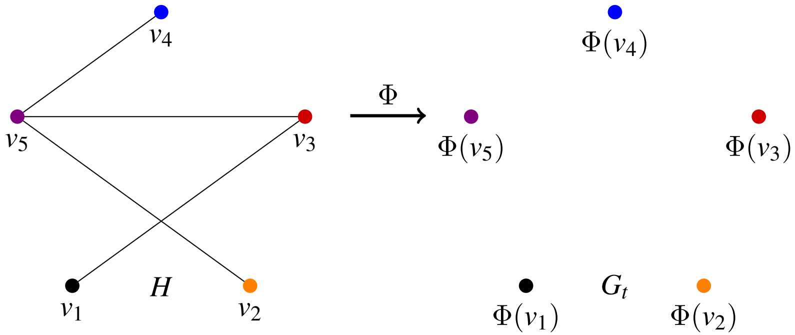

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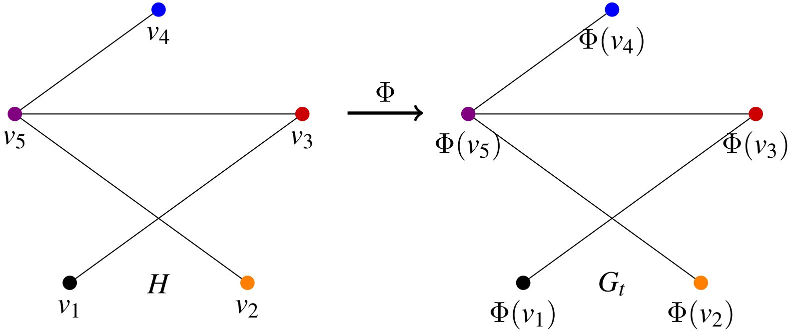

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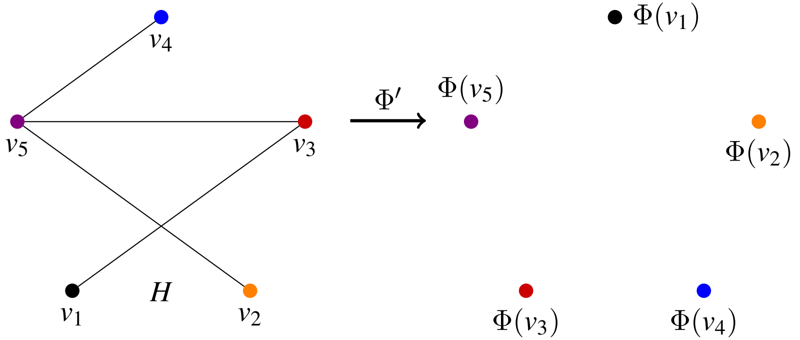

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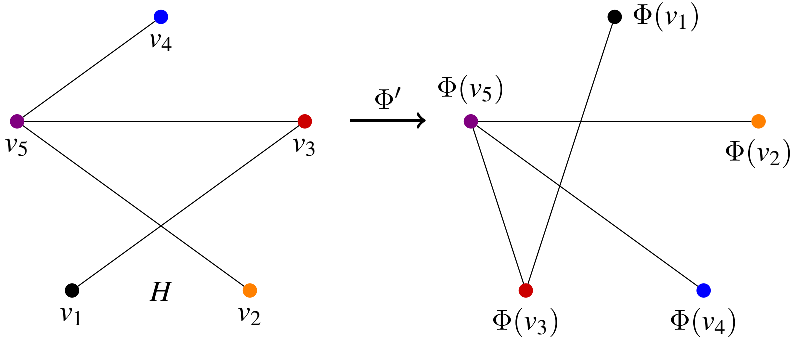

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Lemma

Whp $|F(t_0, \Phi)| \leq \exp\left(-\Delta^{1-\epsilon}\right) n$

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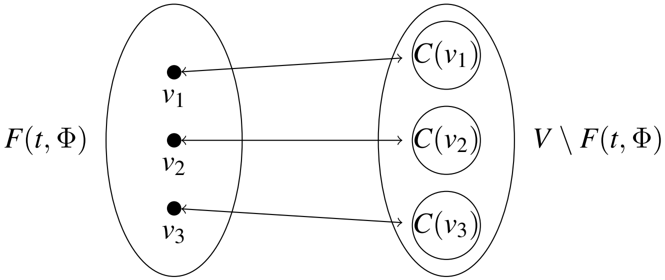

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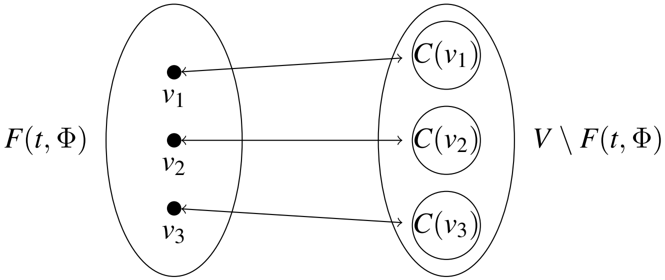

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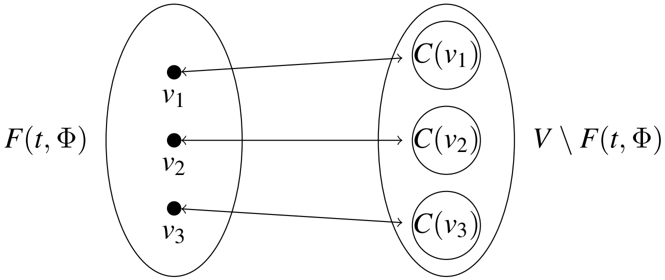

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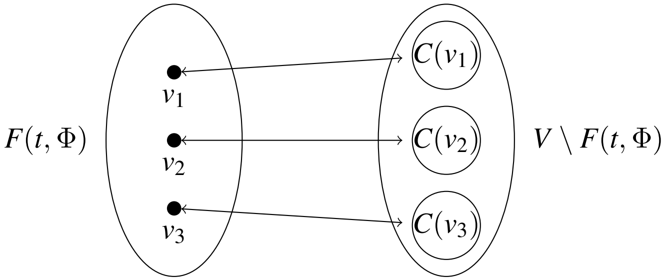

- Update embedding by swapping all possible vertices

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Lemma

There exists an embedding Φ, s.t whp $\quad F(t_1, \Phi) = \emptyset$

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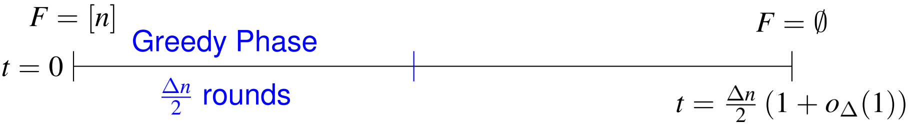

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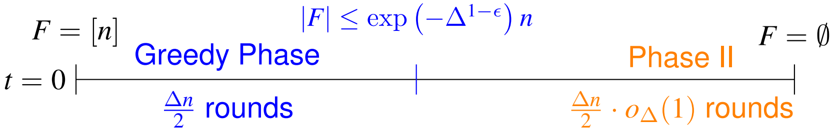

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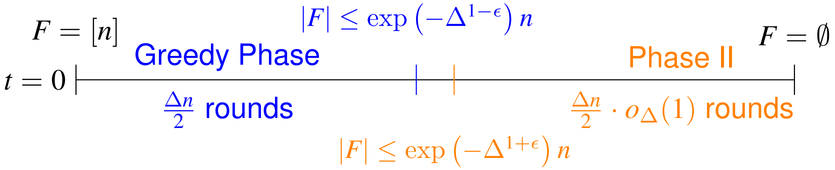

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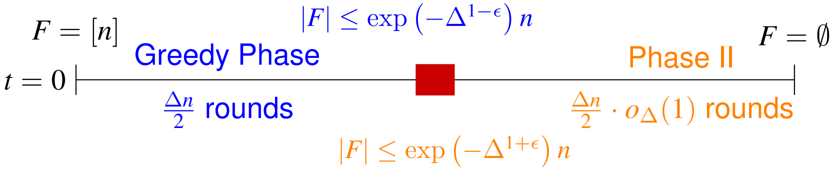

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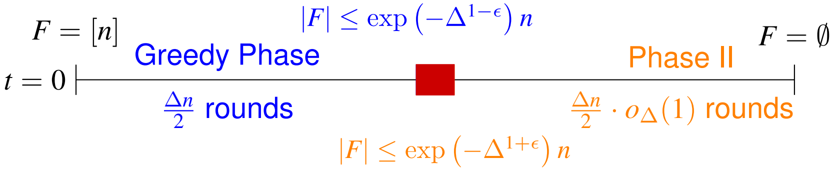

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- Are there properties, for which semirandom **star** process is more efficient than semirandom **tree** process?