# Explosive appearance of cores and bootstrap percolation on lattices

Lyuben Lichev, IST Austria

# Preliminaries on cores

*k*-core of *G*: its largest subgraph with minimum degree *k*.

Easy to construct: iteratively delete vertices of smaller degree.

For $k \geq 3$, Łuczak showed in 1991 that the appearance of a *k*-core in the random graphs process undergoes a discontinuous phase transition.

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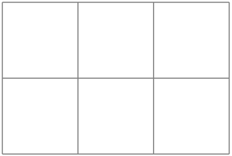
# Our process

Ambient space: $n \times n$ torus in



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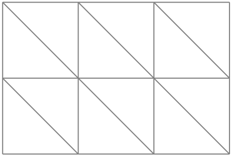

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Object of interest: $k$-core of the graph induced by the present vertices.

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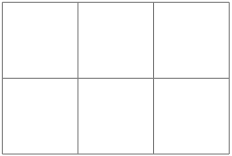

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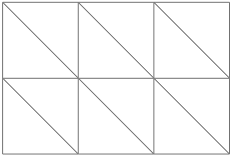

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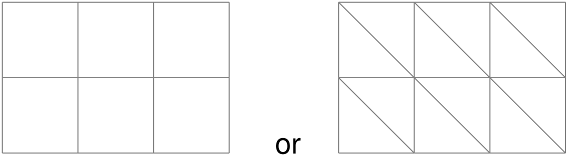


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# Examples

On the square lattice:

- the 2-core emerges as a union of (possibly intersecting) short cycles,
- the 4-core emerges when all vertices appeared.

What about the 3-core?

**Theorem (Bollobás, Duminil-Copin, Morris, Smith, 2014)**

*For large $C > 0$ and small $c > 0$, w.h.p.*

- *at time $n^2 - Cn^2/\log n$, the 3-core is empty.*
- *at time $n^2 - cn^2/\log n$, the 3-core contains $n^2 - o(n^2)$ vertices.*

**Theorem (Hartarsky, L., 2024++)**

*W.h.p. the size of the 3-core jumps from 0 to $n^2 - o(n^2)$ in a single step.*

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State space: $\{\circ, \bullet\}^{n \times n}$ ($\circ / \bullet$ =healthy/infected)

Vertex $x$ in state $\circ$ with at least 2 neighbours $\bullet$: becomes $\bullet$

$\bullet$ never becomes $\circ$

On the square lattice: 3-core process $\iff$ 2-neighbour bootstrap percolation in the complement

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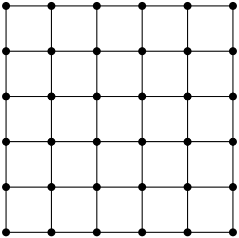

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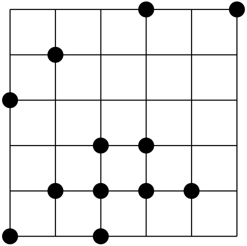

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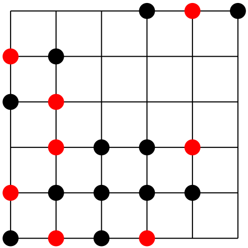

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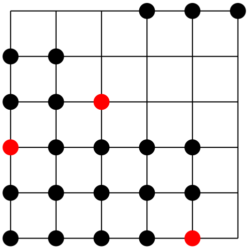

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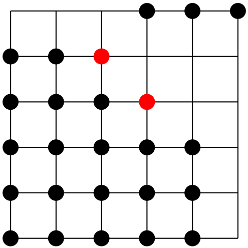

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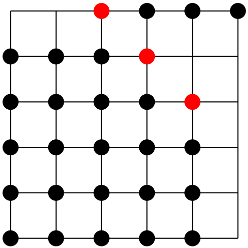

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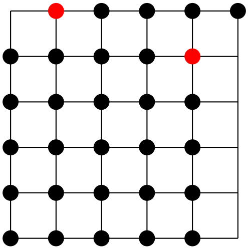

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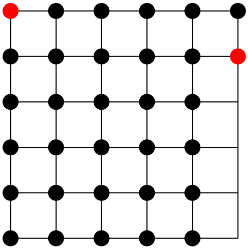

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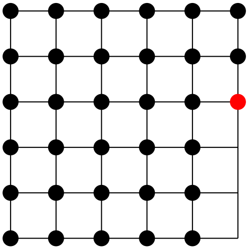

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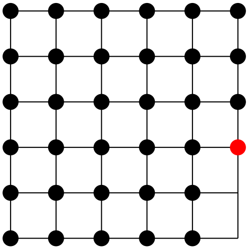

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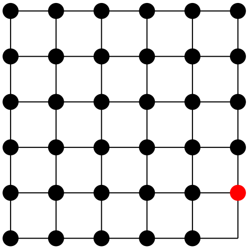

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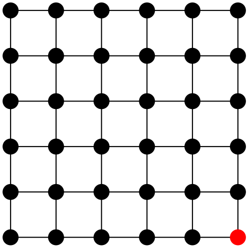

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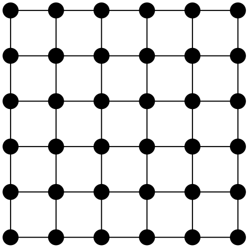

# Why is 2-neighbour bootstrap special?

On the square lattice:

- a single • does infect everyone in 1-neighbour bootstrap (supercritical).
- many finite sets of ○ remain ○ forever in 3-neighbour bootstrap (subcritical).

2-neighbour bootstrap interpolates between the two: a critical model.

**Theorem (Bollobás, Duminil-Copin, Morris, Smith, 2014)**

*In 2-neighbour bootstrap, for large $C > 0$ and small $c > 0$, w.h.p.*

- $Cn^2 / \log n$ *vertices infect the entire torus.*
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**Theorem (Hartarsky, L., 2024++)**

*W.h.p. the size of the infected set grows from $O(n^2 / \log n)$ to $n^2$ in one step.*

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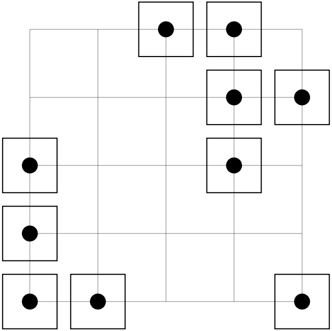

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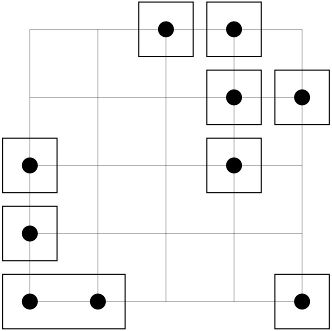

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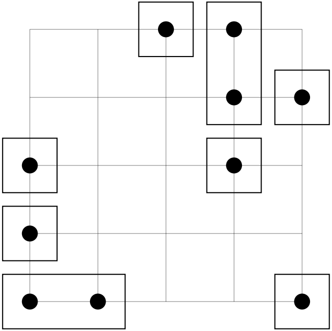

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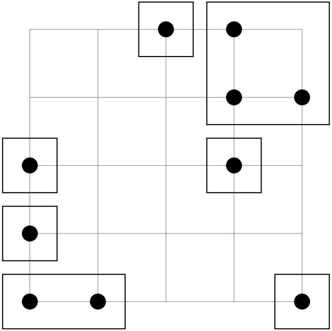

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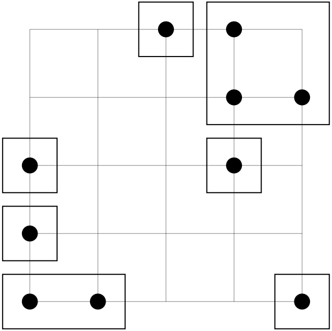

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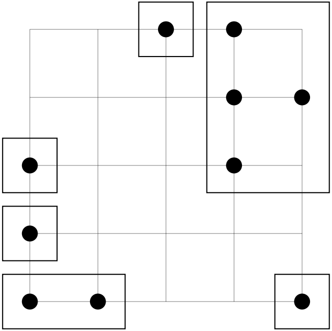

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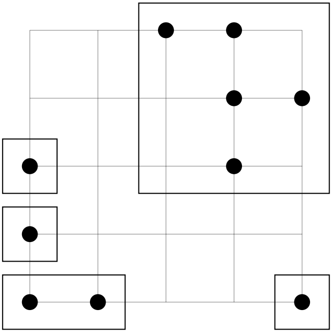

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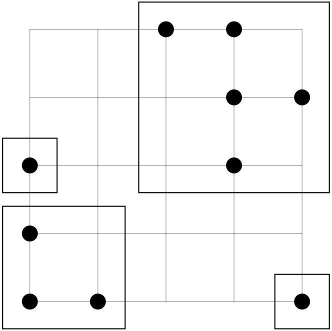

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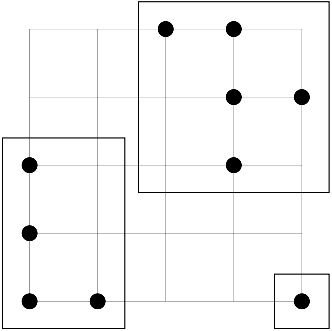

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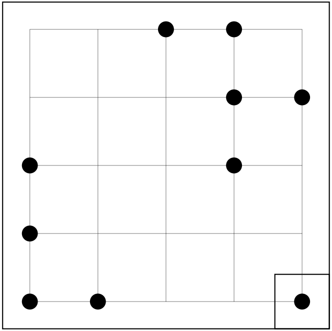

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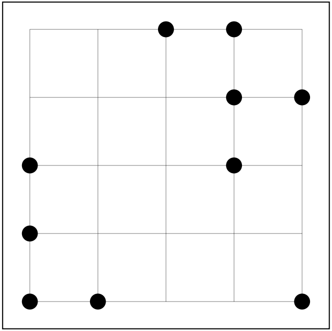

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$\tau =$ smallest time when everyone becomes $\bullet$

Sharp threshold result: $\tau = \Theta(n^2 / \log n)$ w.h.p.

At time $\tau - 1$, a typical box $\Lambda$ of size $D(\log n)^3 \times D(\log n)^3$ whose centre is $\circ$ contains only small groups of $\bullet$-s far from each other.

Conclusion: it stays almost stationary without $\bullet$-s helping from outside.

If its centre becomes $\bullet$ using $\bullet$-s outside $\Lambda$, then a large infected rectangle is found. Such rectangle would spread and infect the entire torus, impossible at time $\tau - 1$.

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We manage to extend our hitting-time result to other critical models.

Fix $K \subseteq \mathbb{R}^2$ convex and invariant under $\pi/2$-rotation, and $\mathcal{K}_s = \mathbb{Z}^2 \cap (sK)$. Choose

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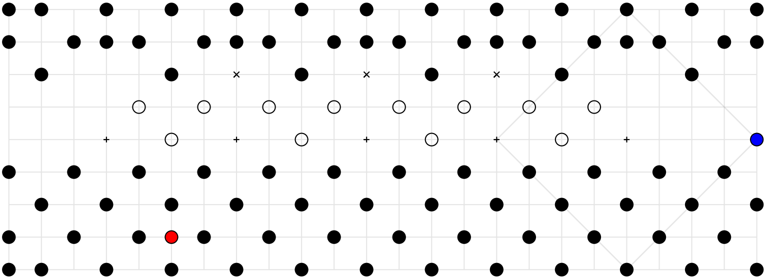

# Thank you for your attention!

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