Counting perfect matchings in Dirac hypergraphs

Yiting Wang

(Joint work with Matthew Kwan and Roodabeh Safavi)

Institute of Science and Technology, Austria

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Overview

Counting Perfect Matchings in Dirac Graphs

Counting perfect matchings in Dirac graphs

- A graph G on n vertices is called *Dirac* if $\delta(G) \geq n/2$.
- By Dirac's theorem, there is at least *one* Hamiltonian cycle in G. When $n \in 2\mathbb{N}$, there are at least two perfect matchings.

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- By Dirac's theorem, there is at least *one* Hamiltonian cycle in G. When $n \in 2\mathbb{N}$, there are at least two perfect matchings.
- In fact, there are *exponentially many* perfect matchings:

Theorem (Sárközy, Selkow and Szemerédi [SSS03])

Let G be a Dirac graph on $n \in 2\mathbb{N}$ vertices and $\Phi(G)$ be the number of perfect matchings in G. Then $\Phi(G) \geq C^{n/2}(n-1)!!$ for some absolute constant $C > 0$.

This is tight up to $\exp(O(n))$ by comparing to $\Phi(K_n)$.

Cuckler and Kahn shows what C should be by establishing the connection with a parameter called "graph entropy".

Graph entropy

Definition (Fractional perfect matching(FPM))

Let $\boldsymbol{x} : E(G) \to [0,1]$ be a weighting on edges. We call $\boldsymbol{x}$ a fractional perfect matching if $\sum_{e: e \ni v} \boldsymbol{x}[e] = 1$ for all vertices $v \in V$.

Definition (Graph entropy)

Given a FPM $\boldsymbol{x}$ in a graph G, the entropy of $\boldsymbol{x}$ is defined as $h(\boldsymbol{x}) = \sum_{e \in E(G)} \boldsymbol{x}[e] \log_2(1/\boldsymbol{x}[e])$. The graph entropy $h(G) := \sup h(\boldsymbol{x})$ where the supremum is taken over all FPMs.

Note that $h(G)$ is efficiently computable because it is the solution of a convex optimization problem.

Counting using entropy

Theorem (Cuckler and Kahn [CK09b, CK09a])

Let G be a Dirac graph on $n \in 2\mathbb{N}$ vertices. Then,
$\Phi(G) = \exp_2(h(G) - (1/2)\log_2 e \cdot n - o(n))$.

- Both a lower and an upper bound.
- It is tight up to $\exp(o(n))$ for *all* Dirac graphs.
- Reduce counting perfect matchings to understanding $h(G)$.

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Theorem (Cuckler and Kahn [CK09b])

Let G be a Dirac graph on $n \in 2\mathbb{N}$ vertices and $p := \delta(G)/(n-1)$. Then,
$\Phi(G) \geq (1 - o(1))^{n/2} \cdot \Phi(K_n) p^{n/2}$.

This result is tight up to $\exp(o(n))$ (for some Dirac graphs).

Counting Perfect Matchings in Dirac hypergraphs

Hypergraph terminology

Let $k \in \mathbb{N}_{\geq 2}$ and $1 \leq d \leq k-1$.

- k-uniform hypergraph G (k-graph)
- minimum d-degree $\delta_d(G)$
- minimum codegree $\delta_{k-1}(G)$
- perfect matchings

In the rest of the talk, we always assume $n \in k\mathbb{N}$.

Dirac hypergraphs

Definition (d-Dirac threshold)

Given $1 \leq d \leq k-1$, let $m_d(k,n)$ is the smallest integer m such that $\delta_d(G) \geq m$ implies G contains a perfect matching. Define the d-Dirac threshold as $\mu_d := \lim_{n\to\infty} m_d(k,n)/\binom{n-d}{k-d}$.

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- The limit exists for all d (Ferber and Kwan [FK22a]).
- $\mu_d = 1/2$ for $3k/8 \leq d \leq k-1$ (Pikhurko [Pik08], Frankl and Kupavskii [FK22b]).
- For $1 \leq d \leq k-1$, there is a precise conjecture on what the value should be (Hàn, Person and Schacht [HPS09], Kühn and Osthus [KO09]).

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Definition (*d-Dirac k-graph*)

For any $1 \leq d \leq k-1$, we call G a *d-Dirac k-graph* if $\delta_d(G) \geq (\mu_d + \gamma)\binom{n-d}{k-d}$ for some constant $\gamma > 0$.

Counting perfect matchings in Dirac hypergraphs

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Theorem (Glock, Gould, Joos, Kühn and Osthus [GGJ$^+$21], Montgomery and Pavez-Signé [MPS23], Kelly, Müyesser and Pokrovskiy [KMP23] Kang, Kelly, Kühn, Osthus and Pfenninger [KKK$^+$22], Pham, Sah, Sawhney and Simkin [PSSS22])

Let $k \in \mathbb{N}$ and $1 \leq d \leq k-1$. Let G be a d-Dirac k-graph on n vertices. Then,

$$\Phi(G) \geq \exp_2((1-1/k)n\log_2 n - Cn)\,,$$

for some absolute constant $C > 0$.

This is tight up to $\exp(O(n))$ by comparing to $\Phi(K_n^{(k)})$.

Counting perfect matchings in Dirac hypergraphs (cont.)

Under the codegree condition, a more precise result is known:

Theorem (Ferber, Krivelevich and Sudakov [FKS14], Ferber, Hardiman and Mond [FHM23])

Let $k \in \mathbb{N}$ and G be a $(k-1)$-Dirac k-graph on n vertices. Then,

$$\Phi(G) \geq (1 - o(1))^{n/k} \cdot \Phi(K_n^{(k)}) p^{n/k},$$

where $p = \delta_{k-1}(G) / \binom{n-k+1}{1}$.

This is tight up to $\exp(o(n))$).

Our Results

Results

Theorem (Kwan, Safavi, W. 24+)

Let $k \in \mathbb{N}_{\geq 2}$ and $1 \leq d \leq k-1$. Let G be a d-Dirac k-graph on n vertices. Then, $\Phi(G) = \exp_2(h(G) - (1 - 1/k)\log_2 e \cdot n - o(n))$.

This result answers if the entropy approach can be extended to hypergraphs asked by Glock, Gould, Joos, Kühn and Osthus [GGJ$^+$21].

Theorem (Kwan, Safavi, W. 24+)

Let $d \geq k/2$. Then, $\Phi(G) \geq (1 - o(1))^{n/k} \cdot \Phi(K_n^{(k)}) p^{n/k}$, where $p := \delta_d(G) / \binom{n-d}{k-d}$.

This results extends the more precise counting result from $d = k - 1$ to $d \geq k/2$.

Proof Sketch

Proof overview

Theorem (Kwan, Safavi, W. 24+)

Let $k \in \mathbb{N}_{\geq 2}$ and $1 \leq d \leq k-1$. Let G be a d-Dirac k-graph on n vertices. Then, $\Phi(G) = \exp_2(h(G) - (1-1/k)\log_2 e \cdot n - o(n))$.

- The upper bound: "entropy method". This part can be deduced from Kahn's work on Sharmir's problem [Kah23].
- The lower bound: "random greedy matching process" with a "good" starting point.

Theorem (Kwan, Safavi, W. 24+)

Let $d \geq k/2$. Then, $\Phi(G) \geq (1-o(1))^{n/k} \cdot \Phi(K_n^{(k)}) p^{n/k}$, where $p \coloneqq \delta_d(G)/\binom{n-d}{k-d}$.

- Construct a large entropy FPM by reducing to the graph case.

Random greedy matching process

Let G be a d-Dirac k-graph and $\boldsymbol{x}^\star$ be the entropy-maximizing FPM in G. Consider the following random process:

- Repeat for $(1 - o(1))n/k$ rounds: remove an edge e with probability proportional to $\boldsymbol{x}^\star[e]$ and $V(e)$.
- Find a perfect matching in the remaining hypergraph.

Question: Isolated vertex? No perfect matching in the remained graph?

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Definition (Well-distributed FPM)

Given a k-graph G, a FPM $\boldsymbol{x}$ in G and a $D \geq 1$, we say $\boldsymbol{x}$ is D-well-distributed if $1/(Dn^{k-1}) \leq \boldsymbol{x}[e] \leq D/n^{k-1}$.

If $\boldsymbol{x}^\star$ is C-well-distributed, then we can keep track of the degree of all subsets of size at most $k-1$, which implies that at the end, the remained graph is still d-Dirac (thanks to the γ-slack in $\delta_d(G)$).

Random greedy matching process (cont.)

Question: How to estimate $\Phi(G)$ via this process?

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The process defines a distribution over (ordered) perfect matchings in G.
We can analyze the entropy of this random variable and use it to lower bound the number of perfect matchings:

Fact (Entropy basics)

Let X be a random perfect matching in G and $H(X)$ be the binary entropy.

- $H(X) \leq \log \Phi(G)$ *and the equality holds if and only if X is a uniform distribution over perfect matchings.*
- $H(X) = H(X_1) + H(X_2 \mid X_1) + \cdots + H(X_{n/k} \mid X_1, \ldots, X_{n/k-1})$.

If $\boldsymbol{x}^\star$ is C-well-distributed, we can keep track of $\sum_{e \in E(i)} \boldsymbol{x}^\star[e]$ and $\sum_{e \in E(i)} \boldsymbol{x}^\star[e] \log(1/\boldsymbol{x}^\star[e])$ then we can lower bound $H(X_i \mid X_1, \ldots, X_{i-1})$.

Find a well-distributed FPM

We do not know if $\boldsymbol{x}^\star$ is C-well-distributed (this is the major difference with the graph case). However, we can find an approximately good one so that the above reasonings still hold approximately.

Lemma (Existence of approximately well-distributed FPM)

Let G be a d-Dirac k-graph and $\varepsilon > 0$. There exists a $O(\varepsilon^{-3k})$-well-distributed FPM $\boldsymbol{x}$ satisfying $h(\boldsymbol{x}^\star) - h(\boldsymbol{x}) \leq \varepsilon n$.

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Step 1: Find a C-well-distributed FPM $\hat{\boldsymbol{x}}$.

- This can be deduced from the existence of spread measure with a simple random adjustment.

Find a well-distributed FPM (cont.)

Step 2: Define $\boldsymbol{x} = (1 - \varepsilon/C) \cdot \boldsymbol{x}^\star + (\varepsilon/C) \cdot \hat{\boldsymbol{x}}$.

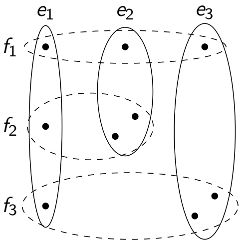
- By construction, $h(\boldsymbol{x}^\star) - h(\boldsymbol{x}) \leq \varepsilon n$ and $\boldsymbol{x}[e] \geq \varepsilon^{-3k}/n^{k-1}$.

Find a well-distributed FPM (cont.)

Step 2: Define $\boldsymbol{x} = (1 - \varepsilon/C) \cdot \boldsymbol{x}^\star + (\varepsilon/C) \cdot \hat{\boldsymbol{x}}$.

- By construction, $h(\boldsymbol{x}^\star) - h(\boldsymbol{x}) \leq \varepsilon n$ and $\boldsymbol{x}[e] \geq \varepsilon^{-3k}/n^{k-1}$.

Step 3: Perform shifting operations on (weighted) shifting structures iteratively (Hàn, Person and Schacht [HPS09]).



$$\boldsymbol{x}[e_1] \cdot \boldsymbol{x}[e_2] \cdot \boldsymbol{x}[e_3] > \boldsymbol{x}[f_1] \cdot \boldsymbol{x}[f_2] \cdot \boldsymbol{x}[f_3]$$

$$(\boldsymbol{x}[e_1] - \Delta) \cdot (\boldsymbol{x}[e_2] - \Delta) \cdot (\boldsymbol{x}[e_3] - \Delta) = (\boldsymbol{x}[f_1] + \Delta) \cdot (\boldsymbol{x}[f_2] + \Delta) \cdot (\boldsymbol{x}[f_3] + \Delta)$$

Lower bound on entropy when $d \geq k/2$

To give a lower bound on $h(G)$ for a d-Dirac k-graph G for $d \geq k/2$, we construct a FPM witnessing it.

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Lemma ([CK09b])

Let G be a bipartite graph on $n + n$ vertices with $\delta(G) \geq n/2$. Then, $h(G) \geq n \log(\delta(G))$.

Consider the following auxiliary bipartite graph G':

- One side contains all size-d subsets, each duplicated $\binom{n}{k-d}$ times and another all size-$(k-d)$ subsets, each duplicated $\binom{n}{d}$ times.
- Put an edge whenever the corresponding d-set and $(k-d)$-set forms an edge.
- Apply lemma in G' to get a FPM and translate it back to G.

Future Directions

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1. We have reduced the problem of counting perfect matchings to understanding the hypergraph entropy. Can we prove a lower bound tight up to $\exp(o(n))$ when $d < k/2$?
2. Our counting results are asymptotic because of the γ slack in the d-degree condition. Can we make them precise by removing the slack?
3. Can we generalize the entropy approach to counting Hamiltonian cycles or other structures?

The end

Thank You for listening! Any question?

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