Branch-Continuous Tree Algebras

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1   Introduction

Algebraic language theory uses tools from algebra to study regular languages. It has been particularly successful in deriving decidable characterisations for various fragments of monadic second-order logic. For instance, a Theorem of Schützenberger [9] states that a regular language is first-order definable if, and only if, its syntactic monoid is aperiodic. The latter condition is decidable as we can compute this syntactic monoid from an automaton for the language and then check it for aperiodicity.

In recent years there has been an effort to extend this algebraic approach to languages of infinite trees. Preliminary results were provided by the group of Bojańczyk [5, 6] with one article considering languages of *regular* trees only, and one considering languages of *thin* trees. The first complete framework that could deal with arbitrary infinite trees was provided by Blumensath [2, 3]. Unfortunately, it turned out to be too complicated and technical for applications.

An interesting new approach has recently been suggested by Blumensath, Bojańczyk, and Klin [4]. They introduced the class of *regular tree algebras* and showed that this class characterises the class of all regular languages of infinite trees in the sense that a tree language is regular if, and only if, it is recognised by such an algebra. Furthermore, they proved the existence of syntactic algebras and showed that these algebras are regular. This is all that is required of a framework if one wants to use it for obtaining decision procedures for subclasses of the regular languages. From a theoretical perspective though, the notion of a regular tree algebra has a serious drawback: the definition is circular in the sense that it is based on the notion of a regular language. Hence, the framework cannot be used as a replacement for other formalisms such as automata or logic, as at least one them is required during the development of the theory of regular tree algebras. It would be very desirable to have an alternative, purely algebraic definition of the notion of a regular tree algebra. Unfortunately, none has been proposed so far.

In this article we introduce a new class of tree algebras, the so-called *branch-continuous tree algebras,* that also characterises the class of regular tree languages. The definition is purely algebraic and therefore does not suffer from the above problems. In particular, branch-continuous algebras seem to be a suitable replacement for automata and logic. On the downside, the new class does not have

the nice closure properties enjoyed by the regular tree algebras. In particular, syntactic algebras are not necessarily branch-continuous. Consequently, regular tree algebras seem to be more suited for practical applications, while the branch-continuous one we consider in the present article appear to be more useful for developing an algebraic theory of regular languages, in particular, as far as the study of algebraic and combinatorial properties of such languages is concerned.

The outline of the article is as follows. We start in Section 2 with some basic definitions including that of a tree algebra. And we begin to develop the algebraic theory of such algebras by introducing some basic notations such as completeness and continuity. In Section 3 we study two completion operations for tree algebras based on the power-set construction. This will pay off later on by allowing several proofs to be quite streamlined and concise.

The heart of this article is Section 4 where we introduce the central notion of a *branch-continuous* tree algebra and we prove that they characterise the class of regular languages. Finally, the last section studies finite representations of branch-continuous tree algebras by introducing an analogue to Wilke algebras in our setting.

2 Tree algebras

One problem with the old framework of Blumensath [2, 3] was the complicated notation for the algebras used. In the meantime a very clean alternative has been proposed which we will adopt for this article. This alternative is based on the category-theoretical notion of a monad and an Eilenberg–Moore algebra. As an example, let us show how define semigroups in this setting.

Given a semigroup $\mathfrak{S} = \langle S, \cdot \rangle$, we can extend the binary product $\cdot : S^2 \to S$ to a product operation $\pi : S^+ \to S$ that takes an arbitrary finite sequence of semigroup elements as argument. Hence, we can formalise semigroups as structures of the form $\langle S, \pi \rangle$ where $\pi : S^+ \to S$ is an associative operation from the *free semigroup* generated by S to S. Associativity in this context means that, given finite sequences $w_0, \ldots, w_{n-1} \in S^+$, we have

$$\pi(\pi(w_0), \ldots, \pi(w_{n-1})) = \pi(w_0 \ldots w_{n-1}).$$

Besides associativity we need one additional axiom, when using a variable-arity product π: we have to require that the product of a single element returns that element.

$$\pi(\langle a \rangle) = a, \quad \text{for } a \in S.$$

Then it follows that every pair $\langle S, \pi \rangle$ satisfying these two axioms corresponds to a semigroup $\langle S, \cdot \rangle$ and vice versa.

This point of view can easily be generalised to other kinds of associative algebras. The only thing we need is the notion of a free algebra generated by some set X. So, suppose we have a functor $\mathbb{T}$ mapping a set X to the free algebra $\mathbb{T}X$ generated by X. Then we can define an algebra as a pair $\langle A, \pi \rangle$ consisting of a set A and a product function $\pi : \mathbb{T}A \to A$. To express our two axioms for such an algebra, we also need functions $\text{flat} : \mathbb{T}\mathbb{T}A \to \mathbb{T}A$ and $\text{sing} : A \to \mathbb{T}A$ that generalise the concatenation and singleton operations

$$(S^+)^+ \to S^+ : \langle w_0, \ldots, w_{n-1} \rangle \mapsto w_0 \ldots w_{n-1}$$
$$S \to S^+ : a \mapsto \langle a \rangle$$

in the semigroup case. Then we can write the associativity axiom as

$$\pi \circ \text{flat} = \pi \circ \mathbb{T}\pi,$$
$$\pi \circ \text{sing} = \text{id}.$$
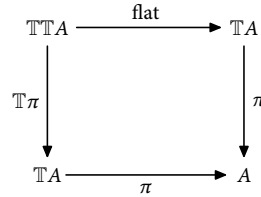
The first of these equations is called the *associative law* for π, the second one the *unit law*.

A pair $\langle A, \pi \rangle$ satisfying these two laws is called a $\mathbb{T}$-*algebra*. For such a $\mathbb{T}$-algebra to be well-behaved, the operations $\mathbb{T}$, flat, and sing should harmonise with each other. As it turns out, three equations are sufficient.

Definition 2.1. Let $\mathcal{C}$ be a category. A triple $\langle \mathbb{T}, \text{flat}, \text{sing} \rangle$ consisting of a functor $\mathbb{T} : \mathcal{C} \to \mathcal{C}$ and two natural transformations $\text{flat} : \mathbb{T} \circ \mathbb{T} \Rightarrow \mathbb{T}$ and $\text{sing} : \text{Id} \Rightarrow \mathbb{T}$ is a *monad* if

$$\text{flat} \circ \text{sing} = \text{id}, \qquad \text{flat} \circ \mathbb{T}\text{sing} = \text{id}, \qquad \text{flat} \circ \mu = \text{flat} \circ \mathbb{T}\text{flat}.$$

$$\begin{array}{ccccc}
\mathbb{T}X & \xrightarrow{\;\text{sing}\;} & \mathbb{T}\mathbb{T}X & \xleftarrow{\;\mathbb{T}\text{sing}\;} & \mathbb{T}X \\
& \searrow^{\text{id}} & \downarrow{\text{flat}} & \swarrow_{\text{id}} & \\
& & \mathbb{T}X & &
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{T}\mathbb{T}\mathbb{T}X & \xrightarrow{\;\text{flat}\;} & \mathbb{T}\mathbb{T}X \\
\downarrow{\mathbb{T}\text{flat}} & & \downarrow{\text{flat}} \\
\mathbb{T}\mathbb{T}X & \xrightarrow{\;\text{flat}\;} & \mathbb{T}X
\end{array}$$
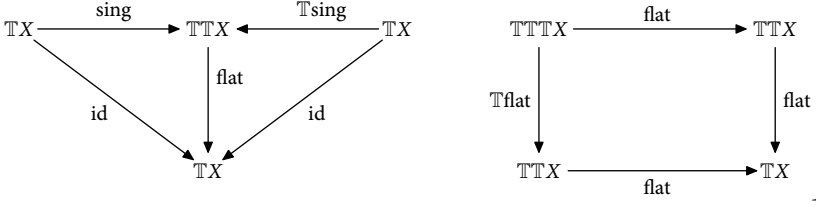
Note that the first and third equation above are just the associative and unit laws for the algebra $\langle \mathbb{T}X, \text{flat}\rangle$. This algebra is called the *free algebra* generated by X.

In our framework, we will adopt this setting of monads and $\mathbb{T}$-algebras. We will use a functor $\mathbb{T}$ mapping a set A to the set of all A-labelled trees, and a flattening operation flat : $\mathbb{T}\mathbb{T}A \to \mathbb{T}A$ that takes a tree labelled by small trees and assembles these into a single large one. Before giving the precise definitions, we need to set up a few preliminaries.

First, we us introduce the category we will be working in. As we have chosen to work with ranked trees and we will be working with ordered algebras, we use the category of ordered and ranked sets.

Definition 2.2. (a) A *ranked set* is a sequence $A = (A_n)_{n<\omega}$ of sets A_n. The members of A_n are called *elements of arity n*. We will tacitly identify such a sequence with its disjoint union $A = \bigcup_n A_n$. This union is equipped with an *arity function* ar : $A \to \omega$ mapping every element $a \in A_n$ to its arity n.

(b) An *ordered set* $\langle A, \leq\rangle$ consists of a ranked set A and a partial order $\leq$ on A such that elements of different arities are incomparable. (Equivalently, we can consider $\leq$ as a sequence $(\leq_n)_n$ where $\leq_n$ is a partial order on A_n.) Usually, we will omit the order $\leq$ from the notation and denote an ordered set just by its domain A.

Definition 2.3. Let A and B be ordered sets and $f : A \to B$ a partial function.
(a) The *domain* of f is the set

$$\mathrm{dom}(f) := \{\, a \in A \mid f(a) \text{ is defined}\,\}.$$

(b) f is *monotone* if

$$a \leq b \quad \text{implies} \quad f(a) \leq f(b), \quad \text{for all } a, b \in \mathrm{dom}(f).$$

(c) f is a *partial function of ordered sets* if it is monotone and it preserves arities. If f is total, we call it a *(total) function of ordered sets*.

(d) pPos denotes the category of all partial functions of ordered sets and Pos the subcategory of all total ones.   ⌟

Remark. The categories pPos and Pos are complete and cocomplete, that is, they have all small limits and colimits. For instance, in the category Pos the *product* $A \times B$ of two ordered sets A and B is given by

$$(A \times B)_n = A_n \times B_n$$

with the component-wise ordering. The *coproduct* $A + B$ is given by the disjoint union

$$(A + B)_n = A_n + B_n$$

where the ordering is induced by those of A and B with elements from different sets being incomparable.

Our functor $\mathbb{T}$ will map a ranked set A to the set of all ranked trees labelled by elements from A. In addition, we will allow leaves of such trees to be labelled with variables $x_0, x_1, x_2, \ldots$ instead. Let us start by defining what we mean by a tree.

Definition 2.4. Let D be a set (unranked).
(a) We denote by $D^{<\omega}$ the set of all finite sequences of elements of D. D^{ω} is the set of all infinite sequences and $D^{\leq\omega} := D^{<\omega} \cup D^{\omega}$. The *empty sequence* is $\langle\rangle$.
(b) The *prefix ordering* on $D^{\leq\omega}$ is

$$x \leq y \quad :\text{iff} \quad y = xz \quad \text{for some } z \in D^{\leq\omega}.$$

If $y = xd$ for $x \in D^{<\omega}$ and $d \in D$, we say that y is an *(immediate) successor* of x and x is an *(immediate) predecessor* of y.
(c) Let $w \in D^{\leq\omega}$. The *length* $|w|$ of w is the ordinal $\alpha \leq \omega$ such that $w \in D^{\alpha}$. We write $w \restriction n$ for the prefix of w of length n and we denote the elements of the sequence w by w_n or by $w(n)$, for $n < |w|$. Thus, $w \restriction n + 1 = (w \restriction n)w_n$.   ⌟

Definition 2.5. Let A be a ranked set.
(a) A *tree domain* is a non-empty set $D \subseteq \omega^{<\omega}$ such that, for all $u \in \omega^{<\omega}$ and $k < \omega$,

- $u \in D$ implies $v \in D$, for all $v \prec u$,
- $uk \in D$ implies $ui \in D$, for all $i < k$.

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(b) An *A-labelled tree* is a function $t : \mathrm{dom}(t) \to A$ where $\mathrm{dom}(t) \subseteq \omega^{<\omega}$ is a tree domain and every vertex $v \in \mathrm{dom}(t)$ has exactly $\mathrm{ar}(t(v))$ immediate successors. We call the number $\mathrm{ar}(v) := \mathrm{ar}(t(v))$ the *arity* of the vertex v.

(c) A *branch* of a tree t is a sequence $\beta \in \omega^{\leq\omega}$ such that $\beta \restriction n \in \mathrm{dom}(t)$, for all finite $n \leq |\beta|$, and $\mathrm{dom}(t)$ contains no successor of β. Hence, a branch β is either finite and $\beta \in \mathrm{dom}(t)$ is a leaf of t, or it is infinite and every proper prefix of β belongs to $\mathrm{dom}(t)$. ⌟

These preliminaries out of the way we can finally define our functor $\mathbb{T}$.

Definition 2.6. Let A be an ordered set. For $n < \omega$, we denote by $\mathbb{T}_n A$ the set of all $(A \cup \{x_0, \ldots, x_{n-1}\})$-labelled trees t where $x_0, \ldots, x_{n-1}$ are new 0-ary symbols and, for every $i < n$, there is at most one vertex $v \in \mathrm{dom}(t)$ labelled by x_i and this vertex is not the root of t. The union is $\mathbb{T}A := \bigcup_n \mathbb{T}_n A$.

Vertices labelled by a variable x_i are called *holes,* or *ports,* with label i. For $t \in \mathbb{T}_n A$, we denote the unique vertex $v \in \mathrm{dom}(t)$ with label x_i by $\mathrm{hole}_i(t)$. If there is no such vertex, we leave $\mathrm{hole}_i(t)$ undefined. The set of all holes is

$$\mathrm{Hole}(t) := \{\, v \in \mathrm{dom}(t) \mid t(v) \in \{x_0, \ldots, x_{n-1}\} \,\}\,. \qquad ⌟$$

To make $\mathbb{T}$ into a functor we also have to define the ordering on $\mathbb{T}A$ and we have to define the operation of $\mathbb{T}$ on functions. The ordering is defined component-wise and $\mathbb{T}f$ applies the function f to all labels. The formal definitions are as follows.

Definition 2.7. (a) For a partial function $f : A \to B$ of ordered sets, we denote by

$$\mathbb{T}f : \mathbb{T}A \to \mathbb{T}B$$

the function that, given a tree $t \in \mathbb{T}_n A$, returns the tree $t' \in \mathbb{T}_n B$ obtained from t by applying f to each label, that is, $\mathrm{dom}(t') = \mathrm{dom}(t)$ and

$$t'(v) = \begin{cases} f(t(v)) & \text{if } t(v) \in A\,, \\ t(v) & \text{if } v \in \mathrm{Hole}(t)\,, \end{cases} \quad \text{for all } v \in \mathrm{dom}(t)\,.$$

We let $\mathbb{T}f(t)$ be undefined, if there is some vertex $v \notin \mathrm{Hole}(t)$ such that $f(t(v))$ is undefined.

(b) Two trees $s \in \mathbb{T}A$ and $t \in \mathbb{T}B$ have *the same shape* if they have the same domains and the same holes (with the same numbering). We denote this relation by $s \simeq_{\mathrm{sh}} t$. We can formally define it by setting

$$s \simeq_{\mathrm{sh}} t \quad :\text{iff} \quad \text{there is some } u \in \mathbb{T}C \text{ and functions } p : C \to A\,, \; q : C \to B$$
$$\text{such that } \mathbb{T}p(u) = s \text{ and } \mathbb{T}q(u) = t\,.$$

(c) For a binary relation $\theta \subseteq A \times B$ and two trees $s \in \mathbb{T}A$ and $t \in \mathbb{T}B$, we write

$$s \; \theta^{\mathbb{T}} \; t \quad \text{:iff} \quad s \simeq_{\text{sh}} t \quad \text{and} \quad s(v) \; \theta \; t(v) \quad \text{for all } v \in \text{dom}(s) \smallsetminus \text{Hole}(s) \,.$$

(d) We consider $\mathbb{T}A$ as an ordered set with order $\leq^{\mathbb{T}}$ where $\leq$ is the order of A. $\lrcorner$

Below we will use relations of the form $\theta^{\mathbb{T}}$ mostly for the ordering $\theta = \leq$ and the membership relation $\theta = \in$. Thus, $\leq^{\mathbb{T}}$ is the componentwise ordering of two trees and $\in^{\mathbb{T}}$ checks that each label of the first tree is an element of the set labelling the corresponding vertex of the second tree.

Lemma 2.8. *The operation $\mathbb{T}$ is a functor* pPos $\to$ pPos. *Its restriction to* Pos *is a functor* Pos $\to$ Pos.

Having found a suitable functor $\mathbb{T}$, we next show that it forms a monad by providing flattening and singleton functions.

Definition 2.9. Let A be an ordered set and $t \in \mathbb{T}\mathbb{T}A$ a tree.
(a) The *flattening function*

$$\text{flat}_A : \mathbb{T}\mathbb{T}A \to \mathbb{T}A$$

maps a tree t to the tree $\text{flat}_A(t) : D \to A$ with domain

$$\begin{aligned}
D := \big\{ \, v_0 \ldots v_{n-1}w \;\big|\; &\text{there is } z \in \text{dom}(t) \text{ such that } |z| = n \,, \\
&w \in \text{dom}(t(z)) \smallsetminus \text{Hole}(t(z)) \text{ and} \\
&v_i = \text{hole}_{z(i)}(t(z \restriction i)) \text{ for } i < n \, \big\}
\end{aligned}$$

and labelling

$$\text{flat}_A(t)(v_0 \ldots v_{n-1}w) := t(z)(w) \,, \quad \text{for } z \in \text{dom}(t) \text{ as above.}$$

(b) The *singleton function*
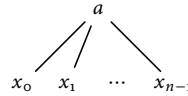
$$\text{sing}_A : A \to \mathbb{T}A$$

maps an element $a \in A_n$ to the tree $t \in \mathbb{T}_n A$ with domain

$$\text{dom}(t) = \{\langle\rangle, \langle 0\rangle, \ldots, \langle n-1\rangle\}$$

and labelling

$$t(v) = \begin{cases} a & \text{if } v = \langle\rangle \,, \\ x_i & \text{if } v = \langle i\rangle \,. \end{cases}$$

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Proposition 2.10. *The functor* $\mathbb{T} : \mathsf{pPos} \to \mathsf{pPos}$ *together with the natural transformations* flat $: \mathbb{T} \circ \mathbb{T} \Rightarrow \mathbb{T}$ *and* sing $: \mathrm{Id} \Rightarrow \mathbb{T}$ *forms a monad. Its restriction to* Pos *also forms a monad.*

Proof. We have to show that flat and sing are natural transformations satisfying the equations

$$\mathrm{flat} \circ \mathrm{sing} = \mathrm{id}\,, \qquad \mathrm{flat} \circ \mathrm{flat} = \mathrm{flat} \circ \mathbb{T}\mathrm{flat}\,.$$

$$\mathrm{flat} \circ \mathbb{T}\mathrm{sing} = \mathrm{id}\,,$$
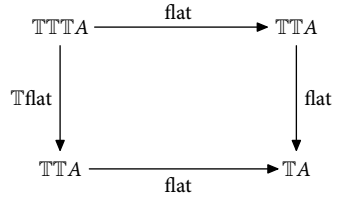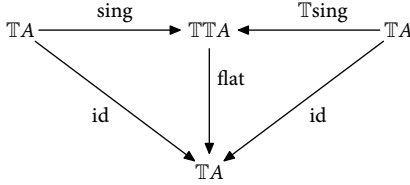
Each of these equations can be established by a straightforward but tedious calculation. $\qquad\square$

After having chosen our monad $\mathbb{T}$, we can introduce the corresponding algebras. For technical reasons, we not only define algebras where the product function is total, but also ones where the product is only pratially defined.

Definition 2.11. (a) A *partial tree algebra* is a $\mathbb{T}$-algebra where we consider $\mathbb{T}$ as a functor on pPos. A *(total) tree algebra* is a $\mathbb{T}$-algebra where we consider $\mathbb{T}$ as a functor on Pos. We use the notation $\mathfrak{A} = \langle A, \pi, \leq \rangle$ for (partial) tree algebras where the ranked set A is the *universe* of $\mathfrak{A}$ and π its *product function*.

(b) A *morphism* $f : \mathfrak{A} \to \mathfrak{B}$ *of partial tree algebras* is a total function $f : A \to B$ of ordered sets that preserves the product, i.e.,

$$f \circ \pi = \pi \circ \mathbb{T}f\,.$$
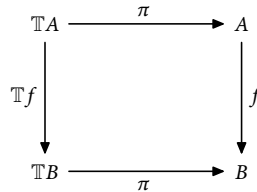
If $\mathfrak{A}$ and $\mathfrak{B}$ are total, we call f a *morphism of total tree algebras.*

We denote the category of all partial tree algebras and their morphisms by pAlg, and that of all total ones by Alg.

To write down finite trees we will use the usual term notation. For instance, $a(b_0, \ldots, b_{n-1})$ denotes the tree with domain $\mathrm{dom}(t) = \{\langle\rangle, \langle 0\rangle, \ldots, \langle n-1\rangle\}$ and labelling

$$t(v) = \begin{cases} a & \text{if } v = \langle\rangle, \\ b_i & \text{if } v = \langle i\rangle. \end{cases}$$

In the motivating example above we have said that the functor $\mathbb{T}$ should map a set to the free algebra generated by it. If $\mathbb{T}$ is a monad, this is automatically the case.

Theorem 2.12. *For each ranked set X, there exists a free tree algebra over X. It has the form $\langle \mathbb{T}X, \mathrm{flat}, \leq^{\mathbb{T}}\rangle$.*

Proof. The fact that $\mathrm{flat} : \mathbb{T}\mathbb{T}X \to \mathbb{T}X$ is the free $\mathbb{T}$-algebra is a standard result in category theory. As the functor $\mathbb{T}$ is a monad, it is left adjoint to the forgetful functor $\mathbb{U} : \mathsf{Alg} \to \mathsf{Pos}$ which maps a tree algebra $\mathfrak{A}$ to its universe A (see, e.g., Proposition 4.1.4 of [7]). Consequently, there exists, for every tree algebra $\mathfrak{A}$ and every function $f : X \to A$, a unique morphism $\varphi : \mathbb{T}X \to \mathfrak{A}$ such that $f = \varphi \circ \mathrm{sing}$. $\square$

Example. Let $\Sigma := \{a, b\}$ where a and b are both binary symbols. Suppose we want to use a morphism $\varphi : \mathbb{T}\Sigma \to \mathfrak{A}$ to recognise the set of all trees $t \in \mathbb{T}\Sigma$ that contain the label a. To do so, we have to remember one bit of information for every input tree t: whether or not t contains an a. So we can attempt to define a tree algebra $\mathfrak{A}$ where for each arity n we have two elements: 0_n and 1_n. Then the product of a tree $s \in \mathbb{T}_n A$ evaluates to 1_n if at least one label in s equals 1_m, and to 0_n otherwise.

Unfortunately, matters are not quite that simple since we have to take products of terms with variables into account. For instance, when multiplying the tree $0_2(x_1, x_2) \in \mathbb{T}_3 A$ we cannot identify the result $a := \pi(0_2(x_1, x_2))$ with the value 0_3 since these two elements behave differently when multiplied:

$$\pi(0_3(1_0, 0_0, 0_0)) = 1_0 \quad \text{and} \quad \pi(a(1_0, 0_0, 0_0)) = \pi(0_2(0_0, 0_0)) = 0_0.$$

That means we have to remember more information about the input tree: which variables is contains. Consequently, we can use for our algebra elements of the form $\langle b, u\rangle$ where $b \in \{0, 1\}$ encodes whether the tree t in question contains an a and $u \subseteq [n]$ is the set of variables of t. The product is then defined in the natural way.

2.1 Completeness and continuity

Let us take a closer look at the interactions between the ordering of a tree algebra and its product. In particular, we are interested in several notions of completeness and continuity.

Definition 2.13. An ordered set A is *complete* if every subset $X \subseteq A_n$, $n < \omega$, has a supremum and an infimum (w.r.t. $\leq$). It is *distributive* if the supremum and infimum operations satisfy the infinite distributive law:

$$\inf_{i \in I} \sup_{k \in K_i} a_{ik} = \sup_{\eta \in \prod_{i \in I} K_i} \inf_{i \in I} a_{i\eta(i)} \,,$$

$$\sup_{i \in I} \inf_{k \in K_i} a_{ik} = \inf_{\eta \in \prod_{i \in I} K_i} \sup_{i \in I} a_{i\eta(i)} \,.$$

⌟

Below we will frequently use morphisms to transfer desirable properties from one tree algebra to another one. The next lemma is a simple example of this technique.

Lemma 2.14. *Let $f : A \to B$ be a surjective function of ordered sets that preserves arbitrary joins.*

  (a) *If A is complete, then so is B.*

  (b) *If A is distributive and φ preserves meets, then B is also distributive.*

Proof. (a) Let $Y \subseteq B_n$. Setting $X := f^{-1}[Y]$ it follows that $f(\sup X) = \sup f[X] = \sup Y$ exists. Hence, every subset of B has a supremum. By a standard argument, this implies that every set also has an infimum. (The infimum of a set is the supremum of its lower bounds.)

(b) Consider elements $b_{ik} \in B$, for $i \in I$ and $k \in K_i$. As φ is surjective, there are elements $a_{ik} \in \varphi^{-1}(b_{ik})$. Consequently,

$$\inf_{i \in I} \sup_{k \in K_i} b_{ik} = \inf_{i \in I} \sup_{k \in K_i} \varphi(a_{ik})$$

$$= \varphi \left(\inf_{i \in I} \sup_{k \in K_i} a_{ik} \right)$$

$$= \varphi \left(\sup_{\eta \in \prod_{i \in I} K_i} \inf_{i \in I} a_{i\eta(i)} \right)$$

$$= \sup_{\eta \in \prod_{i \in I} K_i} \inf_{i \in I} \varphi(a_{i\eta(i)}) = \sup_{\eta \in \prod_{i \in I} K_i} \inf_{i \in I} b_{i\eta(i)} \,. \qquad \square$$

We introduce two notions of continuity: one based on joins and one on meets. For the latter one, we also need a restricted version, where we require continuity only for trees labelled by a given subset of the domain. The two definitions are not entirely symmetric since we are dealing with partial algebras and we want to interpret an undefined result as the least element.

Definition 2.15. Let $\mathfrak{A} = \langle A, \pi, \leq \rangle$ be a partial tree algebra.
(a) $\mathfrak{A}$ is *join-continuous* if we have

$$\pi(t) = \sup \left\{ \pi(s) \mid s \in^{\mathbb{T}} S \text{ and } \pi(s) \text{ is defined} \right\},$$

for all trees $t \in \mathbb{T}A$ and $S \in \mathbb{T}\wp(A)$ such that

$$\pi(t) \text{ is defined} \quad \text{and} \quad t = \mathbb{T}\sup(S).$$

(b) A set $C \subseteq A$ is *meet-continuously embedded* in $\mathfrak{A}$ if, for all $S \in \mathbb{T}\wp(C)$,

$$\pi(\mathbb{T}\inf(S)) = \inf \left\{ \pi(s) \mid s \in^{\mathbb{T}} S \right\},$$

where we require both sides of this equation to be defined for the same trees S and we consider the right-hand side to be defined if every product $\pi(s)$ is defined and the set of these values does have a infimum. The algebra $\mathfrak{A}$ is *meet-continuous* if its universe A is meet-continuously embedded in $\mathfrak{A}$.

(c) We denote by CAlg the subcategory of Alg consisting of all complete, distributive, and join-continuous tree algebras and all morphisms between such algebras that preserve arbitrary joins. ⌋

Again we collect a few technical lemmas that allow us to transfer continuity from one algebra to another.

Lemma 2.16. *Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a surjective morphism of tree algebras that preserves arbitrary joins. If $\mathfrak{A}$ is complete, distributive, and join-continuous, then so is $\mathfrak{B}$.*

Proof. We have already seen in Lemma 2.14 that the algebra $\mathfrak{B}$ is complete and distributive. For join-continuity, consider trees $t \in \mathbb{T}B$ and $S \in \mathbb{T}\wp(B)$ with $t = \mathbb{T}\sup(S)$. We choose some tree $S' \in \mathbb{T}\wp(A)$ with $S = \mathbb{T}\varphi(S')$. Setting $t' = \mathbb{T}\sup(S')$, we obtain

$$t(v) = \sup S(v) = \sup \varphi[S'(v)] = \varphi(\sup S'(v)) = \varphi(t'(v)),$$

for all $v \in \operatorname{dom}(t)$. Therefore,

$$
\begin{aligned}
\pi(t) = \pi(\mathbb{T}\varphi(t')) &= \varphi(\pi(t')) \\
&= \varphi\big(\sup\{\pi(s') \mid s' \in^{\mathbb{T}} S'\}\big) \\
&= \sup\{\varphi(\pi(s')) \mid s' \in^{\mathbb{T}} S'\} \\
&= \sup\{\pi(\mathbb{T}\varphi(s')) \mid s' \in^{\mathbb{T}} S'\} = \sup\{\pi(s) \mid s \in^{\mathbb{T}} S\}.
\end{aligned}
$$

$\square$

Lemma 2.17. *Let* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *be a morphism of complete tree algebras that preserves arbitrary meets. If* $C \subseteq A$ *is meet-continuously embedded in* $\mathfrak{A}$, *then* $\varphi[C]$ *is meet-continuously embedded in* $\mathfrak{B}$.

Proof. Consider trees $t \in \mathbb{T}B$ and $S \in \mathbb{T}\wp(\varphi[C])$ with $t = \mathbb{T}\inf(S)$. We choose some tree $S' \in \mathbb{T}\wp(A)$ with $S = \mathbb{T}\varphi(S')$. Setting $t' = \mathbb{T}\inf(S')$, it follows as in the proof of Lemma 2.16 that

$$
t(v) = \varphi(t'(v)) \quad \text{and} \quad \pi(t) = \inf\{\pi(s) \mid s \in^{\mathbb{T}} S\}. \qquad \square
$$

2.2 Join-generators

Below we will mostly consider tree algebras that are complete and join-continuous. Many properties of such algebras can be reduced to corresponding properties of a subalgebra whose elements generate the full algebra via joins.

Definition 2.18. Let A be an ordered set.
 (a) For a subset $S \subseteq A$, we set

$$
\begin{aligned}
\Downarrow S &:= \{a \in A \mid a \le s \text{ for some } s \in S\}, \\
\Uparrow S &:= \{a \in A \mid a \ge s \text{ for some } s \in S\}.
\end{aligned}
$$

For singletons $S = \{s\}$, we drop the brackets and simply write $\Downarrow s$ and $\Uparrow s$.
 (b) A set $B \subseteq A$ is a set of *join-generators* of A if, for every $a \in A$, there is some set $C \subseteq B$ with $a = \sup C$.

The next lemma summarises some basic properties of sets of join-generators.

Lemma 2.19. *Let* $\mathfrak{A}$ *be a partial tree algebra and* $C \subseteq A$ *a set of join-generators.*

 (a) $a \le b \quad$ iff $\quad c \le a \Rightarrow c \le b$ *for all* $c \in C$.

(b) *If $\varphi, \psi : \mathfrak{A} \to \mathfrak{B}$ are morphisms preserving arbitrary joins, then*

$$\varphi \restriction C = \psi \restriction C \quad \text{implies} \quad \varphi = \psi .$$

(c) *If $\mathfrak{A}$ is join-continuous, then*

$$\pi(t) = \sup \left\{ \pi(s) \mid s \in \mathbb{T}C, \ s \leq^\mathbb{T} t, \ \pi(s) \text{ is defined} \right\}, \quad \text{for all } t \in \mathbb{T}A.$$

Proof. (a) If $a \leq b$, then $c \leq a$ implies $c \leq b$, for all $c \in A$. Conversely, suppose that $c \leq a \Rightarrow c \leq b$, for all $c \in C$. As C is a set of join generators, it follows that

$$a = \sup \{ c \in C \mid c \leq a \} \leq \sup \{ c \in C \mid c \leq b \} = b .$$

(b) Consider an element $a \in A$. Since C is a set of join-generators, we have $a = \sup B$ where $B := \{ c \in C \mid c \leq a \}$. As φ and ψ preserve arbitrary joins, it follows that

$$\varphi(a) = \varphi(\sup B) = \sup \varphi[B] = \sup \psi[B] = \psi(\sup B) = \psi(a) .$$

(c) As $t(\nu) = \sup \{ c \in C \mid c \leq t(\nu) \}$, the claim follows immediately by join-continuity. □

The next remark can be used to simplify proofs that a given morphism preserves meets. It is sufficient to show that it preserves meets of elements of a set of join-generators.

Lemma 2.20. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be complete, distributive tree algebras and $C \subseteq A$ a set of join-generators of $\mathfrak{A}$. If a morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ preserves meets of elements of C and arbitrary joins, it also preserves arbitrary meets.*

Proof. Let $(a_k)_{k \in K}$ be a family of elements of A. We write each $a_k = \sup_{i \in I_k} c_{ki}$ as a join of elements $c_{ki} \in C$. By distributivity, it follows that

$$\begin{aligned}
\varphi\left(\inf_{k \in K} a_k\right) &= \varphi\left(\inf_{k \in K} \sup_{i \in I_k} c_{ki}\right) \\
&= \varphi\left(\sup_{\eta \in \prod_{k \in K} I_k} \inf_{k \in K} c_{k\eta(k)}\right) \\
&= \sup_{\eta \in \prod_{k \in K} I_k} \inf_{k \in K} \varphi(c_{k\eta(k)}) \\
&= \inf_{k \in K} \sup_{i \in I_k} \varphi(c_{ki}) = \inf_{k \in K} \varphi\left(\sup_{i \in I_k} c_{ki}\right) = \inf_{k \in K} \varphi(a_k) .
\end{aligned}$$
□

2.3 Subalgebras

Let us take a look at how a set of join-generators can be embedded in a tree algebra. In particular, we are interested in the case where it induces a subalgebra.

Definition 2.21. Let $\mathfrak{A} = \langle A, \pi, \leq \rangle$ be a partial tree algebra.

(a) A partial tree algebra $\mathfrak{B} = \langle B, \pi', \leq' \rangle$ is a *partial subalgebra* of $\mathfrak{A}$ if $B \subseteq A$ and π' and $\leq'$ are the restrictions of, respectively, π and $\leq$ to the set B, i.e.,

$$\pi'(t) = \begin{cases} \pi(t) & \text{if } \pi(t) \in B, \\ \text{undefined} & \text{otherwise,} \end{cases} \qquad \text{for } t \in \mathbb{T}B,$$

and $\quad a \leq' b \quad$ iff $\quad a \leq b, \qquad\qquad$ for $a, b \in B$.

(b) The *partial subalgebra induced* by a subset $C \subseteq A$ is the partial tree algebra $\mathfrak{A}|_C$ with domain C and product $\pi \upharpoonright D$ where

$$D := \{\, t \in \mathbb{T}C \mid \pi(t) \text{ is defined and } \pi(t) \in C \,\}.$$

(c) The subalgebra *generated* by $C \subseteq A$ is the partial subalgebra with domain

$$\langle C \rangle := \{\, \pi(t) \mid t \in \mathbb{T}C, \, \pi(t) \text{ defined} \,\}.$$

(d) A tree algebra $\mathfrak{A}$ is *finitary* if each domain A_m is finite and there exists a finite set $S \subseteq A$ such that $\langle S \rangle = A$.

Unravelling the definitions we obtain the following criterion for a set inducing a generated subalgebra.

Lemma 2.22. *Let $\mathfrak{A}$ be a partial tree algebra and $C \subseteq A$ a set. The following statements are equivalent:*

(1) $\langle C \rangle = C$

(2) *The inclusion map $i : C \to A$ is a morphism of partial tree algebras.*

(3) $\pi(t) \in C, \quad$ *for every $t \in \mathbb{T}C$ such that $\pi(t)$ is defined.*

Lemma 2.23. *Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a morphism of partial tree algebras and $C \subseteq A$ a set. Then*

$$\varphi[\langle C \rangle] = \langle \varphi[C] \rangle.$$

Proof. We have

$$
\begin{aligned}
\varphi[\langle C \rangle] &= \varphi\big[\,\{\, \pi(t) \mid t \in \mathbb{T}C,\ \pi(t) \text{ defined}\,\}\,\big] \\
&= \{\, \varphi(\pi(t)) \mid t \in \mathbb{T}C,\ \pi(t) \text{ defined}\,\} \\
&= \{\, \pi(\mathbb{T}\varphi(t)) \mid t \in \mathbb{T}C,\ \pi(\mathbb{T}\varphi(t)) \text{ defined}\,\} \\
&= \{\, \pi(t) \mid t \in \mathbb{T}(\varphi[C]),\ \pi(t) \text{ defined}\,\} \\
&= \langle \varphi[C] \rangle\,. \qquad\qquad\qquad\qquad\qquad\qquad\quad\ \square
\end{aligned}
$$

Lemma 2.24. *Let $f : \mathfrak{A} \to \mathfrak{B}$ be a surjective morphism of tree algebras that preserves arbitrary joins. If $C \subseteq A$ is a set of join-generators of $\mathfrak{A}$, then $\varphi[C]$ is a set of join-generators of $\mathfrak{B}$.*

Proof. Let $b \in B$. Since φ is surjective, there is some $a \in A$ with $\varphi(a) = b$. It follows that

$$
\begin{aligned}
b &\geq \sup\{\, d \in \varphi[C] \mid d \leq b\,\} \\
&= \sup\{\, \varphi(c) \mid c \in C,\ \varphi(c) \leq b\,\} \\
&= \varphi\big(\sup\{\, c \in C \mid \varphi(c) \leq \varphi(a)\,\}\big) \\
&\geq \varphi\big(\sup\{\, c \in C \mid c \leq a\,\}\big) = \varphi(a) = b\,. \qquad\qquad \square
\end{aligned}
$$

3 Power-set algebras

Below we will frequently use tree algebras where the elements are subsets of some other tree algebra. In this section we will study a general construction producing such tree algebras. It can be seen as a completion operation for (partial) tree algebras.

3.1 The power-set functor

We start by defining the power-set functor on pPos. Below we will then lift it to a functor on pAlg. In fact, we will define two variant, one for downwards closed sets and one for upwards closed ones.

Definition 3.1. Let A be an ordered set.
(a) The *(downward) power set* $\mathbb{D}A$ of A is the ordered set with domains

$$
\mathbb{D}_n A := \{\, I \subseteq A_n \mid I \text{ is downwards closed}\,\}\,, \quad \text{for } n < \omega\,,
$$

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and ordering

$$I \le J \quad :\text{iff} \quad I \subseteq J, \quad \text{for } I, J \in \mathbb{D}_n A.$$

(b) For a partial function $f : A \to B$ of ordered sets, we define a function $\mathbb{D}f : \mathbb{D}A \to \mathbb{D}B$ by

$$\mathbb{D}f(I) := \Downarrow f[I], \quad \text{for } I \in \mathbb{D}A.$$

(c) For each set A, we define a function $\text{dist}_A : \mathbb{T}(\wp(A)) \to \wp(\mathbb{T}A)$ that maps a tree of sets to a set of trees. The formal definition is

$$\text{dist}_A(t) := \{ s \in \mathbb{T}A \mid s \in^{\mathbb{T}} t \}, \quad \text{for } t \in \mathbb{T}(\wp(A)). \qquad \lrcorner$$

First, let us note that it is straightforward to check that $\mathbb{D}$ forms a monad on pPos.

Proposition 3.2. *The functor $\mathbb{D}$: pPos $\to$ pPos forms a monad where the multiplication* union : $\mathbb{DD}A \to \mathbb{D}A : X \mapsto \bigcup X$ *is given by taking the union and the singleton function* pt $: A \to \mathbb{D}A : a \mapsto \Downarrow\{a\}$ *is the principal ideal operation.*

Example. For every ordered set A, there exists a partial $\mathbb{D}$-algebra $\langle A, \text{sup} \rangle$, where we consider the supremum function as a partial function sup $: \mathbb{D}A \to A$. A partial function $f : A \to B$ preserves arbitrary joins if, and only if, it is a morphism $\langle A, \text{sup} \rangle \to \langle B, \text{sup} \rangle$ of the corresponding $\mathbb{D}$-algebras.

To show that $\mathbb{D}$ lifts to a monad on pAlg, we use a standard technique from category theory based on distributive laws.

Definition 3.3. Let $\langle \mathbb{S}, \mu, \varepsilon \rangle$ and $\langle \mathbb{T}, \nu, \eta \rangle$ be monads. A natural transformation $\lambda : \mathbb{ST} \Rightarrow \mathbb{TS}$ is a *distributive law* if

$$\lambda \circ \mu = \mathbb{T}\mu \circ \lambda \circ \mathbb{S}\lambda, \qquad \lambda \circ \varepsilon = \mathbb{T}\varepsilon,$$
$$\lambda \circ \mathbb{S}\nu = \nu \circ \mathbb{T}\lambda \circ \lambda, \qquad \lambda \circ \mathbb{S}\eta = \eta.$$

$$
\begin{array}{ccccc}
SST A & \xrightarrow{\;\mathbb{S}\lambda\;} & STS A & \xrightarrow{\;\lambda\;} & TSS A \\
{\scriptstyle\mu}\downarrow & & & & \downarrow{\scriptstyle\mathbb{T}\mu} \\
ST A & & \xrightarrow{\qquad\lambda\qquad} & & TS A
\end{array}
\qquad
\begin{array}{ccc}
& TA & \\
{\scriptstyle\varepsilon}\downarrow & & \searrow{\scriptstyle\mathbb{T}\varepsilon} \\
ST A & \xrightarrow{\;\lambda\;} & TS A
\end{array}
$$

$$
\begin{array}{ccccc}
STT A & \xrightarrow{\;\lambda\;} & TST A & \xrightarrow{\;\mathbb{T}\lambda\;} & TTS A \\
{\scriptstyle\mathbb{S}v}\downarrow & & & & \downarrow{\scriptstyle v} \\
ST A & & \xrightarrow{\qquad\lambda\qquad} & & TS A
\end{array}
\qquad
\begin{array}{ccc}
& SA & \\
{\scriptstyle\mathbb{S}\eta}\downarrow & & \searrow{\scriptstyle\eta} \\
ST A & \xrightarrow{\;\lambda\;} & TS A
\end{array}
$$

Lemma 3.4. *The family* $\mathrm{dist} = (\mathrm{dist}_A)_A$ *forms a distributive law* $\mathbb{T}\mathbb{D} \Rightarrow \mathbb{D}\mathbb{T}$.

Proof. First, note that dist_A is a well-defined function $\mathbb{T}\mathbb{D}A \to \mathbb{D}\mathbb{T}A$ since

$$ s \leq^{\mathbb{T}} s' \in^{\mathbb{T}} t \quad \text{implies} \quad s \in^{\mathbb{T}} t, $$

by downwards closure of the sets $t(v)$. Therefore, $\mathrm{dist}_A(t)$ is indeed a downwards closed set of trees. Furthermore, dist_A is obviously monotone.

To see that dist is a natural transformation, let $f : A \to B$ be a partial function of ordered sets. Then

$$
\begin{aligned}
\mathbb{D}\mathbb{T}f\big(\mathrm{dist}_A(t)\big) &= \mathbb{D}\mathbb{T}f\{\, s \mid s \in^{\mathbb{T}} t \,\} \\
&= \Downarrow\big\{\, \mathbb{T}f(s) \mid s \in^{\mathbb{T}} t \,\big\} \\
&= \Downarrow\big\{\, s \mid s \simeq_{\mathrm{sh}} t,\ s(v) \in f[t(v)] \text{ for all } v \,\big\} \\
&= \big\{\, s \mid s \simeq_{\mathrm{sh}} t,\ s(v) \in \Downarrow f[t(v)] \text{ for all } v \,\big\} \\
&= \big\{\, s \mid s \simeq_{\mathrm{sh}} t,\ s(v) \in \mathbb{D}f(t(v)) \text{ for all } v \,\big\} \\
&= \big\{\, s \mid s \in^{\mathbb{T}} \mathbb{T}\mathbb{D}f(t) \,\big\} = \mathrm{dist}_B\big(\mathbb{T}\mathbb{D}f(t)\big).
\end{aligned}
$$

It remains to check the axioms of a distributive law.

$$
\begin{aligned}
(\mathrm{dist} \circ \mathrm{flat})(t) &= \big\{\, s \mid s \in^{\mathbb{T}} \mathrm{flat}(t) \,\big\} \\
&= \big\{\, \mathrm{flat}(s) \mid s \simeq_{\mathrm{sh}} t,\ s(v) \in^{\mathbb{T}} t(v) \text{ for all } v \,\big\} \\
&= \Downarrow\big\{\, \mathrm{flat}(s) \mid s \simeq_{\mathrm{sh}} t,\ s(v) \in^{\mathbb{T}} t(v) \text{ for all } v \,\big\} \\
&= \Downarrow\big\{\, \mathrm{flat}(s) \mid s \in^{\mathbb{T}} t' \,\big\} \quad \text{where } t'(v) = \{\, r \mid r \in^{\mathbb{T}} t(v) \,\} \\
&= \mathbb{D}\mathrm{flat} \circ \big\{\, s \mid s \in^{\mathbb{T}} \mathbb{T}\mathrm{dist}(t) \,\big\} \\
&= \mathbb{D}\mathrm{flat} \circ \mathrm{dist} \circ \mathbb{T}\mathrm{dist},
\end{aligned}
$$

(Note that the second step above relies on the fact that each hole appears at most once in t. This is actually the only place where we need this assumption.)

$$
\begin{aligned}
(\text{dist} \circ \mathbb{T}\text{union})(t) &= \left\{ s \mid s \in^{\mathbb{T}} \mathbb{T}\text{union}(t) \right\} \\
&= \left\{ s \mid s \simeq_{\text{sh}} t,\ s(v) \in \bigcup t(v) \text{ for all } v \right\} \\
&= \Downarrow\!\left\{ s \mid s \simeq_{\text{sh}} t,\ s(v) \in \bigcup t(v) \text{ for all } v \right\} \\
&= \Downarrow\!\left\{ s \mid s \simeq_{\text{sh}} t,\ s(v) \in r(v) \text{ for some } r(v) \in t(v) \right\} \\
&= \Downarrow\!\left\{ s \mid s \in^{\mathbb{T}} r \text{ for some } r \in^{\mathbb{T}} t \right\} \\
&= \bigcup \Downarrow\!\left\{ \left\{ s \mid s \in^{\mathbb{R}} r \right\} \mid r \in^{\mathbb{T}} t \right\} \\
&= \bigcup \Downarrow\!\left\{ \text{dist}(r) \mid r \in^{\mathbb{T}} t \right\} \\
&= \bigcup \mathbb{D}\text{dist}\{ r \mid r \in^{\mathbb{T}} t \} = \text{union} \circ \mathbb{D}\text{dist} \circ \text{dist}\,,
\end{aligned}
$$

$$
\begin{aligned}
(\text{dist} \circ \text{sing})(I) &= \{ s \mid s \in^{\mathbb{T}} \text{sing}(I) \} \\
&= \{ \text{sing}(a) \mid a \in I \} \\
&= \Downarrow\!\{ \text{sing}(a) \mid a \in I \} = \mathbb{D}\text{sing}(I)\,,
\end{aligned}
$$

$$
\begin{aligned}
(\text{dist} \circ \mathbb{T}\text{pt})(t) &= \{ s \mid s \in^{\mathbb{T}} \mathbb{T}\text{pt}(t) \} \\
&= \{ s \mid s \simeq_{\text{sh}} t,\ s(v) \in \Downarrow t(v) \text{ for all } v \} \\
&= \{ s \mid s \leq^{\mathbb{T}} t \} = \Downarrow\!\{t\} = \text{pt}(t)\,. \qquad \square
\end{aligned}
$$

We can use distributive laws to lift a monad from the base category to the category of algebras. The following result can be found, e.g., in Section 9.2 of [1].

Theorem 3.5. *Let* $\langle \mathbb{S}, \mu, \varepsilon \rangle$ *and* $\langle \mathbb{T}, \nu, \eta \rangle$ *be monads and* $\lambda : \mathbb{S}\mathbb{T} \Rightarrow \mathbb{T}\mathbb{S}$ *a distributive law.*

(a) *The composition* $\mathbb{T}\mathbb{S}$ *forms a monad where multiplication and singleton operation are given by the morphisms*

$$
\nu \circ \mathbb{T}\mathbb{T}\mu \circ \mathbb{T}\lambda : \mathbb{T}\mathbb{S}\mathbb{T}\mathbb{S} \Rightarrow \mathbb{T}\mathbb{S} \quad \text{and} \quad \eta \circ \varepsilon : \text{Id} \Rightarrow \mathbb{T}\mathbb{S}\,.
$$

(b) *One can lift* $\mathbb{T}$ *to a functor on* $\mathbb{S}$-*algebras that maps an* $\mathbb{S}$-*algebra* $\pi : \mathbb{S}A \to A$ *to the* $\mathbb{S}$-*algebra* $\mathbb{T}\pi \circ \lambda : \mathbb{S}\mathbb{T}A \to \mathbb{T}A$.

Using this theorem and the distributive law dist we can lift the functor $\mathbb{D}$ to a functor on tree algebras.

Theorem 3.6. *We can lift* $\mathbb{D} : \mathsf{pPos} \to \mathsf{pPos}$ *to a functor* $\mathsf{pAlg} \to \mathsf{pAlg}$ *that maps a partial* $\mathbb{T}$*-algebra* $\mathfrak{A} = \langle A, \pi \rangle$ *to the total* $\mathbb{T}$*-algebra* $\mathbb{D}\mathfrak{A}$ *with domain* $\mathbb{D}A$ *and product*

$$\pi(t) := (\mathbb{D}\pi \circ \mathrm{dist})(t)$$
$$= \left\{\, a \in A \mid a \le \pi(s) \text{ for some } s \in^{\mathbb{T}} t \text{ such that } \pi(s) \text{ is defined} \,\right\},$$

for $t \in \mathbb{T}\mathbb{D}A$.

Proof. We have seen in Lemma 3.4 that there is a distributive law $\mathrm{dist} : \mathbb{T}\mathbb{D} \Rightarrow \mathbb{D}\mathbb{T}$. Therefore, it follows by Theorem 3.5 that we can lift $\mathbb{D}$ to a functor on $\mathbb{T}$-algebras mapping $\pi : \mathbb{T}A \to A$ to the $\mathbb{T}$-algebra with product

$$(\mathbb{D}\pi \circ \mathrm{dist})(t) = \Downarrow\!\left\{\, \pi(s) \mid s \in \mathrm{dist}(t),\ \pi(s) \text{ defined} \,\right\}$$
$$= \Downarrow\!\left\{\, \pi(s) \mid s \in^{\mathbb{T}} t,\ \pi(s) \text{ defined} \,\right\}$$
$$= \left\{\, a \in A \mid a \le \pi(s) \text{ for some } s \in^{\mathbb{T}} t \text{ with } \pi(s) \text{ defined} \,\right\}.$$

$\square$

Using the functor $\mathbb{D}$ we can give a concise definition of join-continuity.

Lemma 3.7. *A tree algebra* $\mathfrak{A} = \langle A, \pi \rangle$ *is join-continuous if, and only if, the supremum function* $\sup : \mathbb{D}\mathfrak{A} \to \mathfrak{A}$ *is a morphism of partial tree algebras.*

Proof. Recall that the product of the algebra $\mathbb{D}\mathfrak{A}$ is given by $\mathbb{D}\pi \circ \mathrm{dist}$. Hence, $\sup$ is a morphism of partial tree algebras if, and only if,

$$\pi \circ \mathbb{T}\sup = \sup \circ (\mathbb{D}\pi \circ \mathrm{dist}).$$

Furthermore, as $\sup X = \sup \Downarrow X$, it is sufficient in the definition of join-continuity, to only consider trees $S \in \mathbb{T}\mathbb{D}A$. Thus, $\mathfrak{A}$ is join-continuous if, and only if, for every $S \in \mathbb{T}\mathbb{D}A$,

$$\pi(\mathbb{T}\sup(S)) = \sup\left\{\, \pi(s) \mid s \in^{\mathbb{T}} S \text{ and } \pi(s) \text{ is defined} \,\right\}.$$

Since

$$(\sup \circ \mathbb{D}\pi \circ \mathrm{dist})(S) = \sup\left\{\, \pi(s) \mid s \in^{\mathbb{T}} S \text{ and } \pi(s) \text{ is defined} \,\right\},$$

the claim follows. $\square$

All tree algebras of the form $\mathbb{D}\mathfrak{A}$ are complete, distributive, and join-continuous.

Proposition 3.8. $\mathbb{D}$ *is a functor of the form* pAlg $\to$ CAlg *where the join and meet in a* $\mathbb{T}$-*algebra* $\mathbb{D}\mathfrak{A}$ *take the form*

$$\sup X = \bigcup X \quad and \quad \inf X = \bigcap X, \quad for\ X \subseteq \mathbb{D}A.$$

Proof. We start by proving that the order of $\mathbb{D}\mathfrak{A}$ is complete and that the joins and meets have the desired form. Let $X \subseteq \mathbb{D}A$. Clearly,

$$\bigcap X \subseteq I \subseteq \bigcup X, \quad \text{for all } I \in X.$$

Furthermore, if

$$K \subseteq I \subseteq L, \quad \text{for all } I \in X,$$

then $K \subseteq \bigcap X$ and $\bigcup X \subseteq L$. Hence, $\bigcap X$ and $\bigcup X$ are the meet and join of X.

Since union and intersection satisfy the infinite distributive law, it further follows that $\mathbb{D}\mathfrak{A}$ is distributive.

Next, we check that every morphism of the form $\mathbb{D}\varphi : \mathbb{D}\mathfrak{A} \to \mathbb{D}\mathfrak{B}$ preserves joins. Let $X \subseteq \mathbb{D}A$.

$$\begin{aligned}
\mathbb{D}\varphi(\bigcup X) &= \{\, b \in B \mid b \le \varphi(a) \text{ for some } a \in \bigcup X \,\} \\
&= \{\, b \in B \mid b \le \varphi(a) \text{ for some } a \in I \text{ with } I \in X \,\} = \bigcup_{I \in X} \mathbb{D}\varphi(I).
\end{aligned}$$

It remains to check join-continuity of $\mathbb{D}\mathfrak{A}$. By Lemma 3.7, it is sufficient to prove that $\sup : \mathbb{D}\mathbb{D}\mathfrak{A} \to \mathbb{D}\mathfrak{A}$ is a morphism of tree algebras, that is,

$$(\mathbb{D}\pi \circ \mathrm{dist}) \circ \mathbb{T}\sup = \sup \circ (\mathbb{D}(\mathbb{D}\pi \circ \mathrm{dist}) \circ \mathrm{dist}).$$

Note that we have shown above that the supremum coincides with the union operation union : $\mathbb{D}\mathbb{D} \Rightarrow \mathbb{D}$, i.e., the multiplication of the monad $\mathbb{D}$. Consequently, we have

$$\begin{aligned}
\sup \circ (\mathbb{D}(\mathbb{D}\pi \circ \mathrm{dist}) \circ \mathrm{dist}) &= \mathrm{union} \circ \mathbb{D}\mathbb{D}\pi \circ \mathbb{D}\mathrm{dist} \circ \mathrm{dist} \\
&= \mathbb{D}\pi \circ \mathrm{union} \circ \mathbb{D}\mathrm{dist} \circ \mathrm{dist} \\
&= \mathbb{D}\pi \circ \mathrm{dist} \circ \mathbb{T}\mathrm{union} = (\mathbb{D}\pi \circ \mathrm{dist}) \circ \mathbb{T}\sup,
\end{aligned}$$

where the second step follows from the fact that union is a natural transformation and the third one from the axioms of a distributive law. $\qquad\square$

Corollary 3.9. *If* $\mathfrak{A} \in \mathsf{CAlg}$, *then* $\sup : \mathbb{D}\mathfrak{A} \to \mathfrak{A}$ *is a morphism of* CAlg.

Proof. By Lemma 3.7, sup is a morphism of pAlg. As $\mathfrak{A}$ is complete, it is a total function. To show that sup preserves joins, let $S \subseteq \mathbb{D}\mathfrak{A}$. Then

$$\sup(\sup S) = \sup \bigcup S = \sup\{\, \sup X \mid X \in S \,\},$$

as desired. $\square$

According to the next proposition, the unit map $A \to \mathbb{D}A$ of the monad $\mathbb{D}$ can be lifted to an embedding $\mathfrak{A} \to \mathbb{D}\mathfrak{A}$ of $\mathbb{T}$-algebras. Hence, we can consider $\mathbb{D}\mathfrak{A}$ as a kind of completion of $\mathfrak{A}$.

Definition 3.10. For a partial tree algebra $\mathfrak{A}$ we define the *canonical embedding* $\eta_{\mathfrak{A}} : \mathfrak{A} \to \mathbb{D}\mathfrak{A}$ by

$$\eta_{\mathfrak{A}}(a) := \Downarrow a, \quad \text{for } a \in A.$$

Proposition 3.11. *The canonical embedding* $\eta_{\mathfrak{A}} : \mathfrak{A} \to \mathbb{D}\mathfrak{A}$ *is a morphism of partial tree algebras preserving meets. Furthermore, the family* $\eta = (\eta_{\mathfrak{A}})_{\mathfrak{A}}$ *is a natural transformation* $\eta : \mathrm{Id} \Rightarrow \mathbb{D}$.

Proof. When considered as a family of morphisms of pPos, the family η is just the singleton operation associated with the monad $\mathbb{D}$. In particular, it is a natural transformation $\mathrm{Id} \Rightarrow \mathbb{D}$. Therefore, it remains to prove that each function $\eta_{\mathfrak{A}}$ is a morphism of partial tree algebras that preserves meets.

We start by checking that $\eta_{\mathfrak{A}}$ commutes with the product π of $\mathfrak{A}$. By Theorem 3.6, the product of $\mathbb{D}\mathfrak{A}$ is the morphism $\mathbb{D}\pi \circ \mathrm{dist}$. Hence, the required equation is

$$(\mathbb{D}\pi \circ \mathrm{dist}) \circ \mathbb{D}\eta = \mathbb{D}\pi \circ \eta = \eta \circ \pi,$$

where the first step follows from the axioms of a distributive law and the second one from the fact that η is a natural transformation.

To see that $\eta_{\mathfrak{A}}$ preserves meets, note that

$$\eta_{\mathfrak{A}}(\inf S) = \{\, c \in A \mid c \le \inf S \,\} = \bigcap \{\, \Downarrow a \mid a \in S \,\} = \bigcap \{\, \eta_{\mathfrak{A}}(a) \mid a \in S \,\}.$$
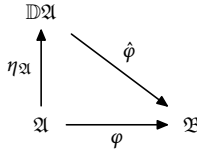
$\square$

3.2 Extension problems

We consider the problem of extending a partial morphism $\mathfrak{A} \to \mathfrak{B}$ to a total one. If the domain of the given morphism is a set of join-generators $C \subseteq A$ and the tree algebra $\mathfrak{B}$ is complete and join-continuous, this poses no problem. In fact, this is equivalent to extend the morphism $\mathfrak{C} \to \mathfrak{B}$ to a morphism $\mathbb{D}\mathfrak{C} \to \mathfrak{B}$.

Proposition 3.12. *For every morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ from an arbitrary partial tree algebra $\mathfrak{A}$ into a complete, join-continuous tree algebra $\mathfrak{B}$, the function*

$$\hat{\varphi} := \sup \circ \mathbb{D}\varphi$$
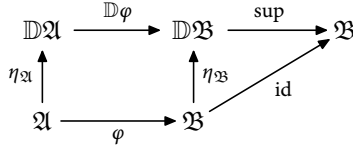
is the unique morphism $\hat{\varphi} : \mathbb{D}\mathfrak{A} \to \mathfrak{B}$ of CAlg *such that*

$$\varphi = \hat{\varphi} \circ \eta_{\mathfrak{A}} \, .$$

Proof. Note that, by definition of the canonical embedding $\eta_{\mathfrak{B}}$, we have

$$(\sup \circ \eta_{\mathfrak{B}})(b) = \sup \Downarrow b = b \, , \quad \text{for } b \in B \, .$$

Thus $\sup : \mathbb{D}\mathfrak{B} \to \mathfrak{B}$ is a left inverse of $\eta_{\mathfrak{B}}$ and

$$\hat{\varphi} \circ \eta_{\mathfrak{A}} = \sup \circ \mathbb{D}\varphi \circ \eta_{\mathfrak{A}} = \sup \circ \eta_{\mathfrak{B}} \circ \varphi = \varphi \, .$$

For uniqueness, suppose that $\psi : \mathbb{D}\mathfrak{A} \to \mathfrak{B}$ is another morphism of CAlg satisfying $\psi \circ \eta_{\mathfrak{A}} = \varphi$. Then

$$\psi \upharpoonright \operatorname{rng} \eta_{\mathfrak{A}} = \varphi = \hat{\varphi} \upharpoonright \operatorname{rng} \eta_{\mathfrak{A}} \, .$$

By Lemma 2.19 (b), this implies that $\psi = \hat{\varphi}$. $\qquad\qquad\square$

In particular, this statement holds for the free algebra.

Theorem 3.13. *Let X be a ranked set, $\mathfrak{T}$ the free algebra over X, and $\mathfrak{A} \in$ CAlg. For every function $f : X \to A$, there exists a unique morphism $\varphi : \mathbb{D}\mathfrak{T} \to \mathfrak{A}$ of CAlg such that*

$$\varphi \circ \eta_{\mathfrak{T}} \circ \mathrm{sing} = f .$$

Proof. The statement can be proved in exactly the same way as Theorem 2.12 by simply replacing the functor $\mathbb{T}$ by $\mathbb{D} \circ \mathbb{T}$. We give an alternative direct proof.

Since $\mathfrak{T}$ is the free algebra of Alg generated by X, there exists a unique morphism $\varphi_0 : \mathfrak{T} \to \mathfrak{A}$ of Alg such that $\varphi_0 \circ \mathrm{sing} = f$. By Proposition 3.12, we can find a unique morphism $\varphi : \mathbb{D}\mathfrak{T} \to \mathfrak{A}$ such that $\varphi \circ \eta_{\mathfrak{T}} = \varphi_0$. Consequently,

$$\varphi \circ \eta_{\mathfrak{T}} \circ \mathrm{sing} = \varphi_0 \circ \mathrm{sing} = f .$$

For uniqueness, suppose that $\psi : \mathbb{D}\mathfrak{T} \to \mathfrak{A}$ is another such morphism. By uniqueness of φ_0,

$$\psi \circ \eta_{\mathfrak{T}} \circ \mathrm{sing} = f \quad \text{implies} \quad \psi \circ \eta_{\mathfrak{T}} = \varphi_0 .$$

By uniqueness of φ, it therefore follows that $\psi = \varphi$. $\qquad\square$

Instead of extending morphisms, we can also consider the problem of extending a partial product $\mathbb{T}B_0 \to B_0$ to a larger set $B \supseteq B_0$. One way to do so is to use a second tree algebra $\mathfrak{A}$ and transfer its product via a given function $A \to B$. This is the content of the following lemma.

Lemma 3.14. *Let $\mathfrak{A} = \langle A, \pi, \leq \rangle$ be a tree algebra and $f : A \to B$ and $\pi' : \mathbb{T}B \to B$ functions of ordered sets such that f is surjective and*

$$f \circ \pi = \pi' \circ \mathbb{T}f .$$

Then $\pi' : \mathbb{T}B \to B$ is a $\mathbb{T}$-algebra.

Proof. For associativity, note that

$$
\begin{aligned}
\pi' \circ \mathbb{T}\pi' \circ \mathbb{T}\mathbb{T}f &= \pi' \circ \mathbb{T}f \circ \mathbb{T}\pi \\
&= f \circ \pi \circ \mathbb{T}\pi \\
&= f \circ \pi \circ \mathrm{flat} \\
&= \pi' \circ \mathbb{T}f \circ \mathrm{flat} = \pi' \circ \mathrm{flat} \circ \mathbb{T}\mathbb{T}f .
\end{aligned}
$$

As f is surjective, so is $\mathbb{T}\mathbb{T}f$. Therefore, it follows that

$$\pi' \circ \mathbb{T}\pi' = \pi' \circ \text{flat}\,.$$

For the unit law, note that

$$\pi' \circ \text{sing} \circ f = \pi' \circ \mathbb{T}f \circ \text{sing} = f \circ \pi \circ \text{sing} = f = \text{id} \circ f\,.$$

By surjectivity of f, it follows that $\pi' \circ \text{sing} = \text{id}$. $\qquad\square$

We aim at extending a product $\pi_0 : \mathbb{T}C \to C$ defined on a set $C \subseteq A$ of join-generators to a join-continuous product $\pi : \mathbb{T}A \to A$. Since the resulting function π has to satisfy Lemma 2.19 (c) it follows that the given product π_0 has to satisfy the following condition.

Definition 3.15. Let $C \subseteq A$ be ordered sets where A is complete. A monotone function $\pi_0 : \mathbb{T}C \to C$ satisfies the *join-extension condition* if, for all trees $S, S' \in \mathbb{T}\mathbb{D}C$,

$$\mathbb{T}\sup(S) = \mathbb{T}\sup(S')$$

implies

$$\sup\{\,\pi_0(s) \mid s \in^{\mathbb{T}} S\,\} = \sup\{\,\pi_0(s') \mid s' \in^{\mathbb{T}} S'\,\}\,. \qquad\lrcorner$$

We need one more technical definition.

Definition 3.16. A partial function $f : A \to B$ of ordered sets is an *embedding of ordered sets* if it is total, injective, and it satisfies

$$a \le b \quad \text{iff} \quad f(a) \le f(b)\,, \quad \text{for all } a, b \in A\,. \qquad\lrcorner$$

Proposition 3.17. *Let $\mathfrak{C} = \langle C, \pi_0, \le \rangle$ be a partial tree algebra and $\varphi : C \to A$ an embedding of ordered sets such that $D := \text{rng}\,\varphi$ is a set of join-generators of A. The image of π_0 under φ satisfies the join-extension condition if, and only if, there exists a unique join-continuous tree algebra $\mathfrak{A} = \langle A, \pi, \le \rangle$ such that $\varphi : \mathfrak{C} \to \mathfrak{A}$ is a morphism of tree algebras. Furthermore, in this case the product $\pi : \mathbb{T}A \to A$ takes the form*

$$\pi(t) = \sup\{\,\pi(s) \mid s \in \mathbb{T}D,\ s \le^{\mathbb{T}} t\,\}\,.$$

Proof. ($\Leftarrow$) Let $\pi_1 : \mathbb{T}D \to D$ be the image of π_0 under φ and let $\pi : \mathbb{T}A \to A$ be a join-continuous extension of π_1. For $S, S' \in \mathbb{T}\mathbb{D}D$ with $\mathbb{T}\sup(S) = \mathbb{T}\sup(S')$ it follows by Lemma 2.19 (c) that
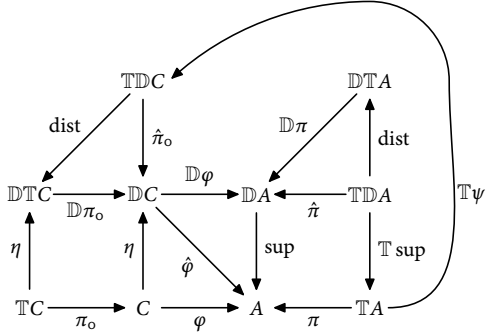
$$\sup \{ \pi_1(s) \mid s \in^{\mathbb{T}} S \} = \pi(\mathbb{T}\sup(S))$$
$$= \pi(\mathbb{T}\sup(S')) = \sup \{ \pi_1(s') \mid s' \in^{\mathbb{T}} S' \}.$$

($\Rightarrow$) We transfer the product of $\mathbb{D}\mathfrak{C}$ to A. Let $\psi : A \to \mathbb{D}C$ be the function defined by

$$\psi(a) := \{ c \in C \mid \varphi(c) \le a \}.$$

Furthermore, we set

$$\hat{\varphi} := \sup \circ \mathbb{D}\varphi,$$
$$\hat{\pi}_0 := \mathbb{D}\pi_0 \circ \mathrm{dist},$$
$$\pi := \hat{\varphi} \circ \hat{\pi}_0 \circ \mathbb{T}\psi,$$
$$\hat{\pi} := \mathbb{D}\pi \circ \mathrm{dist}.$$

Note that, by Theorem 3.6, $\hat{\pi}_0 : \mathbb{T}\mathbb{D}C \to \mathbb{D}C$ and $\hat{\pi} : \mathbb{T}\mathbb{D}A \to \mathbb{D}A$ are the products of the corresponding power-set algebras.

Before proving that $\pi : \mathbb{T}A \to A$ is the desired product, we first show that

$$\hat{\varphi} \circ \hat{\pi}_0 \circ \mathbb{T}(\mathrm{union} \circ \mathbb{D}\psi) = \hat{\varphi} \circ \hat{\pi}_0 \circ \mathbb{T}(\psi \circ \sup).$$

Given $t \in \mathbb{T}\mathbb{D}A$, define

$$S(v) := \{ \varphi(c) \mid \varphi(c) \le a \text{ for some } a \in t(v) \},$$
$$S'(v) := \{ \varphi(c) \mid \varphi(c) \le \sup t(v) \}.$$

As D is a set of join-generators, we have

$$\sup S(v) = \sup \bigcup \{ D \cap {\Downarrow} a \mid a \in t(v) \}$$
$$= \sup \{ \sup(D \cap {\Downarrow} a) \mid a \in t(v) \}$$
$$= \sup \{ a \mid a \in t(v) \} = \sup t(v) = \sup(D \cup {\Downarrow} t(v)) = \sup S'(v).$$

Consequently, it follows from the join-extension condition that

$$(\hat{\varphi} \circ \hat{\pi}_\mathrm{o} \circ \mathbb{T}(\text{union} \circ \mathbb{D}\psi))(t)$$
$$= \sup \big\{ \varphi(\pi_\mathrm{o}(s)) \mid s(v) \in \text{union}(\mathbb{D}\psi(t(v))) \text{ for all } v \big\}$$
$$= \sup \big\{ \varphi(\pi_\mathrm{o}(s)) \mid s(v) \in S(v) \text{ for all } v \big\}$$
$$= \sup \big\{ \varphi(\pi_\mathrm{o}(s')) \mid s'(v) \in S'(v) \text{ for all } v \big\}$$
$$= \sup \big\{ \varphi(\pi_\mathrm{o}(s')) \mid s' \in \mathbb{T}C, \ \varphi(s'(v)) \leq \sup(t(v)) \text{ for all } v \big\}$$
$$= \sup \big\{ \varphi(\pi_\mathrm{o}(s')) \mid s'(v) \in \psi(\sup(t(v))) \text{ for all } v \big\}$$
$$= (\hat{\varphi} \circ \hat{\pi}_\mathrm{o} \circ \mathbb{T}(\psi \circ \sup))(t) .$$

To prove that $\pi : \mathbb{T}A \to A$ is a $\mathbb{T}$-algebra, we apply Lemma 3.14. Thus, we have to check that

$$\pi \circ \mathbb{T}\hat{\varphi} = \hat{\varphi} \circ \hat{\pi}_\mathrm{o} .$$

First, note that, for $I \in \mathbb{D}C$,

$$(\text{union} \circ \mathbb{D}(\psi \circ \varphi))(I) = \bigcup \Downarrow \{ \psi(\varphi(a)) \mid a \in I \}$$
$$= \bigcup \Downarrow \big\{ \{ c \in C \mid \varphi(c) \leq \varphi(a) \} \mid a \in I \big\}$$
$$= \Downarrow \big\{ c \in C \mid \varphi(c) \leq \varphi(a) \text{ for some } a \in I \big\}$$
$$= \Downarrow \big\{ c \in C \mid c \leq a \text{ for some } a \in I \big\}$$
$$= I .$$

Hence, $\text{union} \circ \mathbb{D}(\psi \circ \varphi) = \text{id}$ and it follows that

$$\pi \circ \mathbb{T}\hat{\varphi} = \hat{\varphi} \circ \hat{\pi}_\mathrm{o} \circ \mathbb{T}(\psi \circ \hat{\varphi})$$
$$= \hat{\varphi} \circ \hat{\pi}_\mathrm{o} \circ \mathbb{T}(\psi \circ \sup \circ \mathbb{D}\varphi)$$
$$= \hat{\varphi} \circ \hat{\pi}_\mathrm{o} \circ \mathbb{T}(\psi \circ \sup) \circ \mathbb{T}\mathbb{D}\varphi$$
$$= \hat{\varphi} \circ \hat{\pi}_\mathrm{o} \circ \mathbb{T}(\text{union} \circ \mathbb{D}\psi) \circ \mathbb{T}\mathbb{D}\varphi$$
$$= \hat{\varphi} \circ \hat{\pi}_\mathrm{o} \circ \mathbb{T}(\text{union} \circ \mathbb{D}(\psi \circ \varphi))$$
$$= \hat{\varphi} \circ \hat{\pi}_\mathrm{o} .$$

For join-continuity, it is sufficient by Lemma 3.7 to check that $\sup : \mathbb{D}\mathfrak{A} \to \mathfrak{A}$ is

a morphism of tree-algebras.

$$\begin{aligned}
\sup \circ \hat{\pi} &= \sup \circ \mathbb{D}\pi \circ \mathrm{dist} \\
&= \sup \circ \mathbb{D}(\hat{\varphi} \circ \hat{\pi}_\circ \circ \mathbb{T}\psi) \circ \mathrm{dist} \\
&= \sup \circ \mathbb{D}(\sup \circ \mathbb{D}\varphi) \circ \mathbb{D}(\mathbb{D}\pi_\circ \circ \mathrm{dist})) \circ \mathbb{D}\mathbb{T}\psi \circ \mathrm{dist} \\
&= \sup \circ \mathbb{D}\sup \circ \mathbb{D}\mathbb{D}(\varphi \circ \pi_\circ) \circ \mathbb{D}\mathrm{dist} \circ \mathrm{dist} \circ \mathbb{T}\mathbb{D}\psi \\
&= \sup \circ \mathrm{union} \circ \mathbb{D}\mathbb{D}(\varphi \circ \pi_\circ) \circ \mathbb{D}\mathrm{dist} \circ \mathrm{dist} \circ \mathbb{T}\mathbb{D}\psi \\
&= \sup \circ \mathbb{D}(\varphi \circ \pi_\circ) \circ \mathrm{union} \circ \mathbb{D}\mathrm{dist} \circ \mathrm{dist} \circ \mathbb{T}\mathbb{D}\psi \\
&= \hat{\varphi} \circ \mathbb{D}\pi_\circ \circ \mathrm{dist} \circ \mathbb{T}\mathrm{union} \circ \mathbb{T}\mathbb{D}\psi \\
&= \hat{\varphi} \circ \hat{\pi}_\circ \circ \mathbb{T}(\mathrm{union} \circ \mathbb{D}\psi) \\
&= \hat{\varphi} \circ \hat{\pi}_\circ \circ \mathbb{T}(\psi \circ \sup) \\
&= \pi \circ \mathbb{T}\sup \,,
\end{aligned}$$

where we have used the above claim, the fact that $\sup : \mathbb{D}A \to A$ is a morphism of $\mathbb{D}$-algebras, and that $\mathrm{dist} : \mathbb{T}\mathbb{D} \Rightarrow \mathbb{D}\mathbb{T}$ is a distributive law.

Finally, for uniqueness, suppose that there is another product $\pi' : \mathbb{T}A \to A$ such that $\langle A, \pi', \le \rangle$ is join-continuous and $\varphi : \mathfrak{C} \to \mathfrak{A}$ a morphism. Then it follows by Lemma 2.19 (c) that

$$\begin{aligned}
\pi'(t) &= \sup \left\{ \pi'(s) \;\middle|\; s \in \mathbb{T}D, \; s \le^{\mathbb{T}} t \right\} \\
&= \sup \left\{ \pi'(\mathbb{T}\varphi(s)) \;\middle|\; s \in \mathbb{T}C, \; \mathbb{T}\varphi(s) \le^{\mathbb{T}} t \right\} \\
&= \sup \left\{ \varphi(\pi_\circ(s)) \;\middle|\; s \in \mathbb{T}C, \; \mathbb{T}\varphi(s) \le^{\mathbb{T}} t \right\} \\
&= \sup \left\{ \pi(\mathbb{T}\varphi(s)) \;\middle|\; s \in \mathbb{T}C, \; \mathbb{T}\varphi(s) \le^{\mathbb{T}} t \right\} \\
&= \sup \left\{ \pi(s) \;\middle|\; s \in \mathbb{T}D, \; s \le^{\mathbb{T}} t \right\} \\
&= \pi(t) \,. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Box
\end{aligned}$$

3.3 Upwards closed sets

If we use upwards closed sets instead of downwards closed ones, we obtain a dual version of the power-set operation. Actually, we will slightly break this duality by changing the behaviour of the new functor on non-total functions. The reason for this is the fact that we would like to treat undefined values as least elements.

Definition 3.18. Let $\mathfrak{A}$ be a partial tree algebra.

(a) The *(upward) power-set algebra* $\mathbb{U}\mathfrak{A}$ of $\mathfrak{A}$ has domains

$$\mathbb{U}_n A := \{\, I \subseteq A_n \mid I \text{ is upwards closed} \,\}, \quad \text{for } n < \omega,$$

ordering

$$I \leq J \quad :\text{iff} \quad I \supseteq J, \quad \text{for } I, J \in \mathbb{U}_n A,$$

and product

$$\pi(t) := \Uparrow\{\, \pi(s) \mid s \in^{\mathbb{T}} t \,\},$$

where $\pi(t)$ remains undefined if one of the products $\pi(s)$ is undefined.

(b) For a partial function $f : A \to B$ and a set $I \in \mathbb{U}A$, we define a function $\mathbb{U}f : \mathbb{U}A \to \mathbb{U}B$ by

$$\mathbb{U}f(I) := \Uparrow f[I], \quad \text{if } f(a) \text{ is defined for all } a \in I.$$

Otherwise, $\mathbb{U}f(I)$ remains undefined. ⌟

On sets the functor $\mathbb{U}$ behaves dually to $\mathbb{D}$ in the sense that

$$\mathbb{U}A = (\mathbb{D}(A^{\mathrm{op}}))^{\mathrm{op}}, \quad \text{for } A \in \mathsf{pPos},$$

where $^{\mathrm{op}} : \mathsf{pPos} \to \mathsf{pPos}$ is the functor reversing the order of each set. But note that the corresponding equation for functions does not hold. Still, using this relationship most proofs and results for $\mathbb{D}$ transfer to $\mathbb{U}$ with minor changes. In the following we will therefore omit most of the proofs and only point out the differences.

Proposition 3.19. $\mathbb{U} : \mathsf{pAlg} \to \mathsf{Alg}$ *is a functor mapping partial tree algebras to tree algebras that are complete, distributive, and meet-continuous, and mapping morphisms to morphisms that preserve arbitrary meets. The join and meet of $\mathbb{U}\mathfrak{A}$ are given by*

$$\sup X = \bigcap X \quad \text{and} \quad \inf X = \bigcup X, \quad \text{for } X \subseteq \mathbb{U}A,$$

and the product is given by

$$\pi(t) = (\mathbb{U}\pi \circ \mathrm{dist})(t) = \{\, a \in A \mid a \geq \pi(s) \text{ for some } s \in^{\mathbb{T}} t \,\},$$

for $t \in \mathbb{T}\mathbb{U}A$ such that $\pi(s)$ is defined for all $s \in^{\mathbb{T}} t$.

Proof. As above, the main part of the proof consists in showing that dist forms a distributive law $\mathbb{T}\mathbb{U} \Rightarrow \mathbb{U}\mathbb{T}$. Most steps in the proof of Lemma 3.4 immediately transfer to $\mathbb{U}$. Let us take a closer look at two parts where we need adjustments.

First, to see that dist is a natural transformation, note that

$$\mathbb{U}\mathbb{T}f(\mathrm{dist}_A(t)) \text{ is defined}$$

iff $\quad f(a)$ is defined for all $a \in t(v)$ and $v \in \mathrm{dom}(t)$

iff $\quad \mathbb{U}f(t(v))$ is defined for all $v \in \mathrm{dom}(t)$

iff $\quad \mathbb{T}\mathbb{U}f(t(v))$ is defined .

Furthermore, if these expressions are defined then

$$\mathbb{U}\mathbb{T}f(\mathrm{dist}_A(t)) = \mathbb{U}\mathbb{T}f(\mathrm{dist}_A(t)) = \mathrm{dist}_B(\mathbb{T}\mathbb{U}f(t)) = \mathrm{dist}_B(\mathbb{T}\mathbb{U}f(t)) .$$

It remains to check the axioms of a distributive law. Note that $\mathbb{U}f(I) = (\mathbb{D}f(I^{\mathrm{op}}))^{\mathrm{op}}$, provided that $\mathbb{U}f(I)$ is defined. Once we have shown that the expressions on both sides are defined on the same inputs, we can therefore use duality and the equations for the functor $\mathbb{D}$ to prove the corresponding axioms for $\mathbb{U}$. Note that the only functions the functor $\mathbb{U}$ is applied to in these axioms are flat, dist, and sing, which are all total. Hence, both sides of the equations are defined for all inputs. □

Lemma 3.20. *Let $\mathfrak{A}$ be a tree algebra. A subset $C \subseteq A$ is meet-continuously embedded in $\mathfrak{A}$ if, and only if, the infimum function* $\inf : \mathbb{U}C \to \mathfrak{A}$ *is a morphism of partial tree algebras.*

Proof. As the product of the algebra $\mathbb{U}\mathfrak{A}$ is given by $\mathbb{U}\pi \circ \mathrm{dist}$, it follows that inf is a morphism of partial tree algebras if, and only if,

$$\pi \circ \mathbb{T}\inf = \inf \circ (\mathbb{U}\pi \circ \mathrm{dist}) .$$

Again, in the definition of meet-continuity it is sufficient to only consider trees $S \in \mathbb{T}\mathbb{U}C$. Thus, C is meet-continuously embedded in $\mathfrak{A}$ if, and only if, for every $S \in \mathbb{T}\mathbb{U}C$,

$$\pi(\mathbb{T}\inf(S)) = \inf \{ \, \pi(s) \mid s \in^{\mathbb{T}} S \, \} ,$$

where we use the convention that the right-hand side is defined if, and only if, $\pi(s)$ is defined for all $s \in^{\mathbb{T}} S$. Since

$$(\inf \circ \mathbb{U}\pi \circ \mathrm{dist})(S) = \inf \{ \, \pi(s) \mid s \in^{\mathbb{T}} S \, \}$$

(with the same convention), the claim follows. □

Proposition 3.21. *The canonical embedding*

$$\zeta_{\mathfrak{A}} : \mathfrak{A} \to \mathbb{U}\mathfrak{A} : a \mapsto \Uparrow a$$

is a morphism of partial tree algebras that preserves joins.

Proposition 3.22. *For every morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ from an arbitrary partial tree algebra $\mathfrak{A}$ into a complete, meet-continuous tree algebra $\mathfrak{B}$, the function*

$$\hat{\varphi} := \inf \circ \mathbb{U}\varphi$$

is the unique morphism $\hat{\varphi} : \mathbb{U}\mathfrak{A} \to \mathfrak{B}$ such that $\hat{\varphi}$ preserves meets and

$$\varphi = \hat{\varphi} \circ \zeta_{\mathfrak{A}} .$$

Definition 3.23. Let $C \subseteq A$ be ordered sets where A is complete. A monotone function $\pi_0 : \mathbb{T}C \to C$ satisfies the *meet-extension condition* if, for all trees $S, S' \in \mathbb{T}\mathbb{U}C$,

$$\mathbb{T}\inf(S) = \mathbb{T}\inf(S')$$

implies

$$\inf \{\, \pi_0(s) \mid s \in^{\mathbb{T}} S \,\} = \inf \{\, \pi_0(s') \mid s' \in^{\mathbb{T}} S' \,\} \,,$$

where we again regard each side of this equation to be defined if, and only if, the products are defined for all trees $s \in^{\mathbb{T}} S$ and $s' \in^{\mathbb{T}} S'$, respectively. ⌐

Recall the definition of an embedding of ordered sets from Definition 3.16.

Proposition 3.24. *Let $\mathfrak{C} = \langle C, \pi_0, \leq \rangle$ be a partial tree algebra, $\varphi : C \to A$ an embedding of ordered sets, and let $B \subseteq A$ be the closure of $D := \operatorname{rng} \varphi$ under meets. The image of π_0 under φ satisfies the meet-extension condition if, and only if, there exists a unique meet-continuous tree algebra $\mathfrak{B} = \langle B, \pi, \leq \rangle$ such that $\varphi : \mathfrak{C} \to \mathfrak{B}$ is a morphism of tree algebras. Furthermore, in this case the product $\pi : \mathbb{T}B \to B$ takes the form*
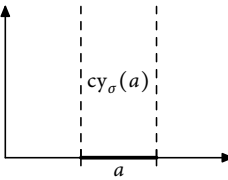
$$\pi(t) = \inf \{\, \pi(s) \mid s \in \mathbb{T}D , \, s \geq^{\mathbb{T}} t \,\} \,.$$

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4 Branch-continuous algebras

4.1 Semigroup-like algebras and traces

Our next aim is to develop a structure theory for tree algebras that are generated in a certain way by an ω-semigroup. Such tree algebras will be the central notion of our framework. In this section, we collect a bit of technical material needed for this task. We start by noting that every tree algebra comes with canonical embeddings $A_m \to A_n$, for $m \le n$.

Definition 4.1. Let $\mathfrak{A}$ be a tree algebra and $\sigma : [m] \to [n]$ an injective function with $m \le n < \omega$. The σ-*cylinder* over an element $a \in A_m$ is

$$\mathrm{cy}_\sigma(a) := a(x_{\sigma(0)}, \dots, x_{\sigma(m-1)}) \in A_n.$$

In the special case where $\mathrm{ar}(a) = 1$, we also use the short hand

$$\mathrm{cy}_k(a) := a(x_k) = \mathrm{cy}_\sigma(a), \quad \text{where } \sigma : [1] \to [n] : 0 \mapsto k.$$

A further tool we will need is the unravelling operation. To define it, we need a notion of 'which variables actually appear in a label $a \in A$'. For this reason we introduce what we call *cylindrical structures*.

Definition 4.2. Let A be an ordered set.

(a) A *cylindrical structure* of A is a function associating with every element $a \in A$ a pair $\langle a^\circ, \sigma_a \rangle$ consisting of an element $a^\circ \in A$ with $\mathrm{ar}(a^\circ) \le \mathrm{ar}(a)$ and a strictly increasing function $\sigma_a : [\mathrm{ar}(a^\circ)] \to [\mathrm{ar}(a)]$. We require that

- $a \le b \quad \Rightarrow \quad a^\circ \le b^\circ$ and $\sigma_a = \sigma_b$,

- $(a^\circ)^\circ = a^\circ$ and $\sigma_{a^\circ} = \mathrm{id}$.

(b) A cylindrical structure on A is *compatible* with a product $\pi : \mathbb{T}A \to A$ if

$$a = \mathrm{cy}_{\sigma_a}(a^\circ), \quad \text{for all } a \in A.$$

(c) The *unravelling* of a tree $t \in \mathbb{T}A$ with respect to a given cylindrical structure on A is the tree

$$\mathrm{un}(t) := \mathrm{flat}(S),$$

where $S \simeq_{\mathrm{sh}} t$ is defined by

$$S(v) := a^{\circ}(x_{\sigma_a(0)}, \ldots, x_{\sigma_a(\mathrm{ar}(a^{\circ})-1)}) \in \mathbb{T}_{\mathrm{ar}(a)}A, \quad \text{for } a := t(v).$$

Note that, for trees, the unravelling operation is rather simple. It only reorders the successors of the vertices and removes unreachable subtrees.

Lemma 4.3. $\mathrm{un} : \mathbb{T}A \to \mathbb{T}A$ *is an idempotent morphism of tree algebras.*

Proof. Monotonicity of un follows from the first condition in the definition of a cylindrical structure, and the fact that $\mathrm{un}(\mathrm{un}(t)) = \mathrm{un}(t)$ from the second one. Hence it remains to prove that un commutes with the product flat of $\mathbb{T}A$. Let $t \in \mathbb{T}\mathbb{T}A$ and let $S \in \mathbb{T}\mathbb{T}\mathbb{T}A$ be a tree such that

$$\mathrm{un}(t(v)) = \mathrm{flat}(S(v)), \quad \text{for } v \in \mathrm{dom}(t).$$

This implies that $S' := \mathrm{flat}(S)$ is the tree such that

$$\mathrm{un}(\mathrm{flat}(t)) = \mathrm{flat}(S').$$

Consequently,

$$\mathrm{flat}(\mathbb{T}\mathrm{un}(t)) = \mathrm{flat}(\mathbb{T}\mathrm{flat}(S)) = \mathrm{flat}(\mathrm{flat}(S)) = \mathrm{un}(\mathrm{flat}(t)). \qquad \square$$

Lemma 4.4. *Let $\mathfrak{A}$ be a partial tree algebra whose universe A is equipped with a cylindrical structure that is compatible with the product of $\mathfrak{A}$ and such that all cylinder maps cy_σ are defined.*

$$\pi(\mathrm{un}(t)) = \pi(t), \quad \text{for all } t \in \mathrm{dom}(\pi).$$

Proof. Note that (with the notation of the definition above)

$$t(v) = \mathrm{cy}_{\sigma_{t(v)}}(t(v)^{\circ}) = \pi(S(v)) \quad \text{implies} \quad t = \mathbb{T}\pi(S).$$

Hence, $\pi(t) = \pi(\mathbb{T}\pi(S)) = \pi(\mathrm{flat}(S)) = \pi(\mathrm{un}(t))$. $\qquad \square$

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Corollary 4.5. *Let $\mathfrak{A}$ be a partial tree algebra whose universe A is equipped with a cylindrical structure that is compatible with the product of $\mathfrak{A}$ and such that all cylinder maps cy_σ are defined and such that $\mathrm{rng\, un} \subseteq \mathrm{dom}\,\pi$. The function $\hat{\pi} := \pi \circ \mathrm{un}$ is the unique total function $\mathbb{T}A \to A$ that extends the product π of $\mathfrak{A}$ and such that $\widehat{\mathfrak{A}} := \langle A, \hat{\pi} \rangle$ is a total tree algebra.*

Proof. We set $\hat{\pi} := \pi \circ \mathrm{un}$. By the preceding lemma, this is the only possible extension of π. To see that it in fact defines a tree algebra, note that

$$\hat{\pi} \circ \mathrm{flat} = \pi \circ \mathrm{un} \circ \mathrm{flat} = \pi \circ \mathrm{flat} \circ \mathbb{T}\mathrm{un}$$
$$= \pi \circ \mathbb{T}\pi \circ \mathbb{T}\mathrm{un}$$
$$= \pi \circ \mathbb{T}\hat{\pi}$$
$$\subseteq \pi \circ \mathrm{un} \circ \mathbb{T}\hat{\pi} = \hat{\pi} \circ \mathbb{T}\hat{\pi}\,,$$

where the second but last step follows from (a). Since $\hat{\pi}$ is total, the two sides of this inclusion are equal and $\hat{\pi}$ is the product of a tree algebra. $\square$

Below we will be interested in ways an ω-semigroup can sit inside a tree algebra and in tree algebras generated by some ω-semigroup they contain. The basic building blocks we will use in this context are subalgebras of the following form.

Definition 4.6. A partial tree algebra $\mathfrak{A}$ is *semigroup-like* if $A = \langle A_0 \cup A_1 \rangle$. ⌟

Note that, given a tree algebra $\mathfrak{A}$, every subalgebra of the form $\langle S \rangle$, for a set $S \subseteq A_0 \cup A_1$, is semigroup-like. In order to study semigroup-like tree algebras and to relate them to the tree algebras they are contained in, we introduce the notion of a *trace* of a tree, which intuitively corresponds to the product of t along *a single branch*. A trace along a given branch β of t, is a path-shaped tree u whose labels are point-wise greater or equal to the corresponding labels of the vertices of β. The formal definition is as follows.

Definition 4.7. Let $\mathfrak{A}$ be a complete tree algebra, $\mathfrak{S} \subseteq \mathfrak{A}$ a semigroup-like subalgebra, $t \in \mathbb{T}_m A$ a tree, and β a branch of t.
   (a) We denote by $\mathrm{cl}(S)$ the closure of S under non-empty meets.
   (b) An *S-trace* of t along β is a tree $u \in \mathbb{T}_m(S_0 \cup S_1)$ such that

$$\mathrm{dom}(u) = \left\{ \mathsf{o}^n \mid n < \omega,\ n \le |\beta| \right\}$$

$$\mathrm{cy}_{\beta(n)}(u(\mathsf{o}^n)) \ge t(\beta \upharpoonright n)\,, \quad \text{for all } n < |\beta|,$$

and $\quad u(o^n) \geq t(\beta), \quad$ if $n := |\beta|$ is finite.

(c) An *S-quasi-trace* of t along β is a tree $u \in \mathbb{T}_m(S \cup \{\top\})$ such that $t \leq^{\mathbb{T}} u$ and, for every $n < |\beta|$, either

$$u(\beta \restriction n) = \top \quad \text{or} \quad u(\beta \restriction n) = \mathrm{cy}_{\beta(n)}(a), \quad \text{for some } a \in S_0 \cup S_1.$$

Note that the unravelling of a quasi-trace is a trace. By Lemma 4.4 it further follows that every product in a semigroup-like tree algebra reduces to the product along some branch. The following result collects this and a few other character-isations of semigroup-like tree algebras.

Lemma 4.8. *Let $\mathfrak{A}$ be a partial tree algebra such that the domain $\mathrm{dom}(\pi) \subseteq \mathbb{T}A$ of the product is closed under all cylinder maps $\mathrm{cy}_\sigma : \mathbb{T}_m A \to \mathbb{T}_n A$. The following statements are equivalent.*

(1) *$\mathfrak{A}$ is semigroup-like.*

(2) *Every element $a \in A$ is of the form $a = \mathrm{cy}_\sigma(b)$, for some $b \in A_0 \cup A_1$ and some injective function σ.*

(3) *For every tree $t \in \mathbb{T}A$ such that $\pi(t)$ is defined, there exists a tree $u \in \mathbb{T}(A_0 \cup A_1)$ with $\pi(u) = \pi(t)$.*

(4) *Every tree $t \in \mathrm{dom}(\pi)$ has an A-trace u with $\pi(u) = \pi(t)$.*

Proof. (2) $\Rightarrow$ (1) Let $a \in A$. Then $a = \mathrm{cy}_\sigma(b)$, for some $b \in A_0 \cup A_1$ and some σ. Hence, $a = \mathrm{cy}_\sigma(b) \in \langle A_0 \cup A_1 \rangle$.

(1) $\Rightarrow$ (3) $\pi(t) \in A = \langle A_0 \cup A_1 \rangle$ implies that $\pi(t) = \pi(u)$, for some tree $u \in \mathbb{T}(A_0 \cup A_1)$.

(3) $\Rightarrow$ (2) Given an element $a \in A_n$, we can use (3) to find a tree $u \in \mathbb{T}_n(A_0 \cup A_1)$ such that $a = \pi(\mathrm{sing}(a)) = \pi(u)$. We distinguish two cases. If u does not contain a variable, then $u = \mathrm{cy}_\sigma(u')$ for some $u' \in \mathbb{T}_0(A_0 \cup A_1)$ and $\sigma : \varnothing \to [n]$. Consequently, we have $\pi(u) = \pi(\mathrm{cy}_\sigma(u')) = \mathrm{cy}_\sigma(\pi(u'))$. Since $\pi(u') \in A_0$, the claim follows.

If u does contain a variable x_k, then $u = \mathrm{cy}_\sigma(u')$ where $\sigma(o) = k$ and $u' \in \mathbb{T}_1(A_0 \cup A_1)$ is the tree obtained from u by replacing x_k by x_0. As above, it follows that $\pi(u) = \mathrm{cy}_k(\pi(u'))$ and $\pi(u') \in A_1$.

(4) $\Rightarrow$ (3) is trivial since an A-trace is an element of $\mathbb{T}(A_0 \cup A_1)$.

(2) $\Rightarrow$ (4) Let $u := \mathrm{un}(t)$ with the cylindrical structure given by (2). Then u is an A-trace of t and Lemma 4.4 implies that $\pi(t) = \pi(u)$. $\qquad\square$

In some cases, the product of a tree is determined by the products of its S-traces, or its S-quasi-traces. We start by transforming traces into quasi-traces.

Lemma 4.9. *Let $\mathfrak{A}$ be a complete tree algebra, $\mathfrak{S} \subseteq \mathfrak{A}$ a semigroup-like subalgebra, $t \in \mathbb{T}A$ a tree, and β a branch of t. For every S-trace u of t along β, there exists an S-quasi-trace $\hat{u}$ of t along β with $\pi(\hat{u}) = \pi(u)$.*

Proof. Let u be an S-trace of t along β. We define $\hat{u} \simeq_{\mathrm{sh}} t$ by

$$
\hat{u}(v) := \begin{cases} \top & \text{if } v \nleq \beta, \\ \mathrm{cy}_{\beta(n)}(u(\mathrm{o}^n)) & \text{if } v = \beta \restriction n < \beta, \\ u(\mathrm{o}^{|\beta|}) & \text{if } v = \beta. \end{cases}
$$

Then $\hat{u}$ is an S-quasi-trace, $\mathrm{un}(\hat{u}) = u$, and it follows by Lemma 4.4 that $\pi(\hat{u}) = \pi(u)$. $\qquad\square$

Lemma 4.10. *Let $\mathfrak{A}$ be a complete tree algebra, $\mathfrak{S} \subseteq \mathfrak{A}$ a semigroup-like subalgebra, and $t \in \mathbb{T}A$.*

$$
\begin{aligned}
\pi(t) &\le \inf \left\{ \pi(u) \mid u \text{ an } S\text{-quasi-trace of } t \right\} \\
&\le \inf \left\{ \pi(u) \mid u \text{ an } S\text{-trace of } t \right\}.
\end{aligned}
$$

Proof. The first inequality follows since we have $t \le^{\mathbb{T}} u$, for every S-quasi-trace u. The second inequality follows by Lemma 4.9. $\qquad\square$

In the important special case of a meet-continuously embedded subalgebra, the above inequalities become equalities.

Proposition 4.11. *Let $\mathfrak{A}$ be a tree algebra, $\mathfrak{S} \subseteq \mathfrak{A}$ a semigroup-like subalgebra that is meet-continuously embedded in $\mathfrak{A}$, and set $C := \mathrm{cl}(S)$. For every tree $t \in \mathbb{T}C$,*

$$
\begin{aligned}
\pi(t) &= \inf \left\{ \pi(u) \mid u \text{ an } S\text{-quasi-trace of } t \right\} \\
&= \inf \left\{ \pi(u) \mid u \text{ an } S\text{-trace of } t \right\}.
\end{aligned}
$$

Proof. Let $t \in \mathbb{T}C$ be a tree. We define a tree $U \in \mathbb{T}_n \wp(\langle S \rangle)$ by

$$
U \simeq_{\mathrm{sh}} t \quad \text{and} \quad U(v) := \left\{ c \in \langle S \rangle \mid c \ge t(v) \right\}.
$$

Since every element of C is a non-empty meet of elements of $\langle S \rangle$, we have $t = \mathbb{T} \inf(U)$. Hence, it follows by meet-continuous embeddedness that

$$
\pi(t) = \inf \left\{ \pi(s) \mid s \in^{\mathbb{T}} U \right\}.
$$

By Lemma 4.10, it is therefore sufficient to show that

$$\{\, \pi(s) \mid s \in^{\mathbb{T}} U \,\} \subseteq \{\, \pi(u) \mid u \text{ an } S\text{-trace of } t \,\} \,.$$

Thus, consider a tree $s \in^{\mathbb{T}} U$. We can use Lemma 4.8 to find an S-trace u of s such that $\pi(u) = \pi(s)$. Since $t \leq^{\mathbb{T}} s$, it follows that u is also an S-trace of t. $\qquad\square$

As a first application of traces, we show that the meet-closure of a semigroup-like subalgebra is closed under products.

Proposition 4.12. *Let $\mathfrak{A}$ be a complete tree algebra and $\mathfrak{S} \subseteq \mathfrak{A}$ a semigroup-like subalgebra that is meet-continuously embedded in $\mathfrak{A}$.*

(a) $\mathrm{cl}(S)$ *is meet-continuously embedded in $\mathfrak{A}$.*

(b) $\langle \mathrm{cl}(S) \rangle = \mathrm{cl}(S)$

Proof. (a) Let $t \in \mathbb{T}A$ and $T \in \mathbb{T}\wp(\mathrm{cl}(S))$ be trees with $t = \mathbb{T}\inf(T)$. For each $c \in \mathrm{cl}(S)$, we choose a set $U_c \subseteq S$ with $c = \inf U_c$ and we define trees $R, U \simeq_{\mathrm{sh}} T$ by

$$U(v) := \{\, U_c \mid c \in T(v) \,\} \quad \text{and} \quad R(v) := \bigcup U(v) \,.$$

Then $\inf R(v) = \inf T(v) = t(v)$ and it follows by meet-continuous embeddedness of S that

$$\begin{aligned}
\pi(t) &= \inf \{\, \pi(r) \mid r \in^{\mathbb{T}} R \,\} \\
&= \inf \{\, \pi(r) \mid r \in^{\mathbb{T}} u \text{ for some } u \in^{\mathbb{T}} U \,\} \\
&= \inf \{\, \inf \{\, \pi(r) \mid r \in^{\mathbb{T}} u \,\} \mid u \in^{\mathbb{T}} U \,\} \\
&= \inf \{\, \pi(\mathbb{T}\inf(u)) \mid u \in^{\mathbb{T}} U \,\} = \inf \{\, \pi(s) \mid s \in^{\mathbb{T}} T \,\} \,,
\end{aligned}$$

where the last step follows from the fact that there is a bijective correspondence between trees $u \in^{\mathbb{T}} U$ and $s \in^{\mathbb{T}} T$.

(b) Let $t \in \mathbb{T}(\mathrm{cl}(S))$. By definition of $\mathrm{cl}(S)$, there is a tree $T \in \mathbb{T}(\wp(S) \smallsetminus \{\varnothing\})$ such that $t = \mathbb{T}\inf(T)$. We can use Lemma 4.8 to find, for every tree $s \in^{\mathbb{T}} T$, an S-trace $\hat{s}$ of s such that $\pi(\hat{s}) = \pi(s)$. Furthermore, $\hat{s} \in \mathbb{T}S$ implies that $\pi(s) = \pi(\hat{s}) \in \langle S \rangle = S$. By meet-continuous embeddedness of S in $\mathfrak{A}$, it follows that

$$\pi(t) = \inf \{\, \pi(s) \mid s \in^{\mathbb{T}} T \,\} = \inf \{\, \pi(\hat{s}) \mid s \in^{\mathbb{T}} T \,\}$$

is a non-empty meet of elements in S. Thus, $\pi(t) \in \mathrm{cl}(S)$. $\qquad\square$

4.2 ω-semigroups

Before finally defining the class of algebras we are interested in, let us recall some facts regarding ω-semigroups. We use a definition that facilitates a comparison with tree algebras.

Definition 4.13. (a) The *word functor* $\mathbb{W} : \mathsf{pPos} \to \mathsf{pPos}$ is defined by

$$\mathbb{W}_0 A := A_1^\omega \cup A_1^{<\omega} A_0,$$
$$\mathbb{W}_1 A := A_1^{<\omega},$$
$$\mathbb{W}_n A := \varnothing, \quad \text{for } n > 1.$$

(b) An (*ordered partial*) *ω-semigroup* $\mathfrak{S} = \langle S, \pi \rangle$ is a $\mathbb{W}$-algebra $\pi : \mathbb{W}S \to S$. We use the usual notation for products in ω-semigroups. That is, for elements $a \in S_1$ and $b \in S_0 \cup S_1$, we write $a \cdot b$, or just ab, instead of $a(b)$. Similarly, we write $\prod_{i<n} a_i$ instead of $a_0(\dots(a_{n-1})\dots)$.

We denote the category of all ω-semigroups by SGrp.

(c) A partial ω-semigroup $\mathfrak{S}$ is *meet-continuous* if, for all sequences $w \in \mathbb{W}S$ and $U \in \mathbb{W}\wp(S)$ with $w = \mathbb{W}\inf(U)$, we have

$$\pi(w) = \inf \{ \pi(u) \mid u \in^{\mathbb{W}} U \}.$$

(As usual, $\in^{\mathbb{W}}$ denotes the component-wise element relation and we require that, if one side of the equation is defined, so is the other.) ⌟

Since we can regard words as trees without branching, the word functor $\mathbb{W}$ is some kind of subfunctor of the tree functor $\mathbb{T}$. The lemma below makes this relationship precise.

Definition 4.14. Let $\langle \mathbb{T}_0, \mu_0, \varepsilon_0 \rangle$ and $\langle \mathbb{T}_1, \mu_1, \varepsilon_1 \rangle$ be monads.

(a) A natural transformation $\varphi : \mathbb{T}_0 \Rightarrow \mathbb{T}_1$ is a *morphism of monads* if

$$\varepsilon_1 = \varphi \circ \varepsilon_0 \quad \text{and} \quad \mu_1 \circ (\varphi \circ \mathbb{T}_0 \varphi) = \varphi \circ \mu_0.$$

In this case we say that $\mathbb{T}_0$ is a *reduct* of $\mathbb{T}_1$.

(b) Let $\rho : \mathbb{T}_0 \Rightarrow \mathbb{T}_1$ be a morphism of monads and $\mathfrak{A} = \langle A, \pi, \leq \rangle$ a $\mathbb{T}_1$-algebra. The *ρ-reduct* of $\mathfrak{A}$ is the $\mathbb{T}_0$-algebra $\langle A, \pi \circ \rho_A, \leq \rangle$. ⌟

Lemma 4.15. *There exists a morphism of monads $\rho : \mathbb{W} \Rightarrow \mathbb{T}$ satisfying*

$$\mathrm{dist} \circ \rho = \mathbb{D}\rho \circ \mathrm{dist} \quad \textit{and} \quad \mathrm{dist} \circ \rho = \mathbb{U}\rho \circ \mathrm{dist}$$

(depending on whether we consider dist *as natural transformations $\mathbb{T}\mathbb{D} \Rightarrow \mathbb{D}\mathbb{T}$ and $\mathbb{W}\mathbb{D} \Rightarrow \mathbb{D}\mathbb{W}$, or as $\mathbb{T}\mathbb{U} \Rightarrow \mathbb{U}\mathbb{T}$ and $\mathbb{W}\mathbb{U} \Rightarrow \mathbb{U}\mathbb{W}$).*

Proof. The function $\rho_A : \mathbb{W}A \to \mathbb{T}A$ maps a word $w \in \mathbb{W}A$ to the tree $t \in \mathbb{T}A$ with domain

$$\mathrm{dom}(t) := \{\, \mathrm{o}^n \mid n < |w| \,\}$$

and labelling

$$t(\mathrm{o}^n) := w(n)\,, \quad \text{for } n < |w|\,.$$

It is straightforward to check that ρ is a morphism of monads. For the additional equations, note that

$$\mathrm{dist}(\rho(U)) = \Downarrow\{\, t \mid t \in^{\mathbb{T}} \rho(U) \,\} = \Downarrow\{\, \rho(u) \mid u \in^{\mathbb{W}} U \,\} = \mathbb{D}\rho(\mathrm{dist}(U))\,,$$

and similarly for the functor $\mathbb{U}$. $\qquad\square$

We can associate with every a tree algebra an ω-semigroup as follows.

Definition 4.16. The ω-*semigroup* $\mathrm{SG}(\mathfrak{A})$ *associated* with a partial tree algebra $\mathfrak{A}$ is the ω-semigroup with domains A_0 and A_1 whose product is inherited from that of $\mathfrak{A}$.
$\lrcorner$

Lemma 4.17. $\mathrm{SG} : \mathsf{Alg} \to \mathsf{SGrp}$ *is a functor.*

Conversely, we can associate with every ω-semigroup $\mathfrak{S}$ a semigroup-like tree algebra $\mathrm{TA}(\mathfrak{S})$ which consists of elements of the form a or $a(x_i)$, for an ω-semigroup element a and an optional variable x_i.

Definition 4.18. (a) The *tree algebra* $\mathrm{TA}(\mathfrak{S})$ *associated with* a partial ω-semi-group $\mathfrak{S}$ has domains

$$\mathrm{TA}_n(S) := S_0 \cup (S_1 \times [n])\,, \quad \text{for } n < \omega\,,$$

and the ordering

$$
\begin{aligned}
x \le y \quad :\text{iff} \quad & x, y \in S_0 \text{ and } x \le y \text{ in } \mathfrak{S}\,, \text{ or} \\
& x = \langle a, i \rangle,\ y = \langle b, i \rangle \text{ for } a \le b \text{ and } i < n\,.
\end{aligned}
$$

We will use the more suggestive notation $a(x_k)$ for the elements of the form $\langle a, k \rangle$.

We define the product $\pi(t)$ of a tree $t \in \mathbb{T}(\mathrm{TA}(S))$ as the product of its unravelling $\mathrm{un}(t)$ (which is a tree with a single path) in the ω-semigroup $\mathfrak{S}$. To

make this precise, we need a bit of preparation. Let $i : S \to \mathrm{TA}(S)$ be the natural embedding where

$$i(a) = \begin{cases} a & \text{if } a \in S_0 \,, \\ a(x_0) & \text{if } a \in S_1 \,. \end{cases}$$

We start by defining the cylinder maps $\mathrm{cy}_\sigma : \mathrm{TA}_m(S) \to \mathrm{TA}_n(S)$.

$$\mathrm{cy}_\sigma(a) := \begin{cases} a & \text{if } a \in S_0 \,, \\ b(x_{\sigma(k)}) & \text{if } a = b(x_k) \in S_1 \times [m] \,. \end{cases}$$

For the general case, consider a tree $t \in \mathbb{T}_m(\mathrm{TA}(S))$. We unravel t with respect to the following cylindrical structure. Let $a \in A_n$.

- If $a = b \in S_0$, we set $a^\circ := b \in A_0$ and $\sigma_a : \varnothing \to [n]$.

- If $a = b(x_k) \in S_1 \times [n]$, we set $a^\circ := b(x_0) \in A_1$ and $\sigma_a : [1] \to [n] : 0 \mapsto k$.

Note that the unravelling $\mathrm{un}(t)$ is of the form $\mathrm{un}(t) = \mathrm{cy}_\sigma(s)$ for some $s \in \mathrm{rng}\,\rho$. Fix the word $u \in \mathbb{W}S$ with $s = \rho(\mathbb{W}i(u))$. (u is unique since ρ and i are injective.) We set

$$\pi(t) := \mathrm{cy}_\sigma(i(\pi(u)))\,.$$

(b) For a morphism $\varphi : \mathfrak{S} \to \mathfrak{T}$ of ω-semigroups, we define the function $\mathrm{TA}(\varphi) : \mathrm{TA}(\mathfrak{S}) \to \mathrm{TA}(\mathfrak{T})$ by

$$\mathrm{TA}(\varphi)(a) := \begin{cases} \varphi(a) & \text{if } a \in S_0 \,, \\ \varphi(b)(x_k) & \text{if } a = b(x_k) \in S_1 \times [n] \,. \end{cases}$$

Proposition 4.19. *Let $\mathfrak{S}$ be an ω-semigroup.*

(a) $\mathrm{TA}(\mathfrak{S})$ *is a semigroup-like tree algebra.*

(b) $\mathfrak{S}$ *is meet-continuous if, and only if,* $\mathrm{TA}(\mathfrak{S})$ *is meet-continuous.*

(c) $\mathrm{TA} : \mathsf{SGrp} \to \mathsf{Alg}$ *is a functor.*

Proof. (c) follows from (a) and the definition of TA.

(a) For monotonicity, let $t \le t'$. Using the notation from the definition of the product (with primes where appropriate), it follows that

$$\mathrm{un}(t) \le \mathrm{un}(t') \quad \Rightarrow \quad s \le s' \quad \Rightarrow \quad u \le u'\,.$$

Consequently,

$$\pi(t) = \mathrm{cy}_\sigma(i(\pi(u))) \le \mathrm{cy}_\sigma(i(\pi(u'))) = \pi(t')\,.$$

For the unit law, let $a \in \mathrm{TA}(S)$ and $t := \mathrm{sing}(a)$. If $a \in S_{\mathrm{o}}$, then $u = \langle a \rangle$ and

$$\pi(t) = \mathrm{cy}_\varnothing(i(\pi(u))) = \mathrm{cy}_\varnothing(i(a)) = a\,.$$

If $a = b(x_k) \in S_1 \times [m]$, then $u = \langle b \rangle$ and

$$\pi(t) = \mathrm{cy}_k(i(\pi(u))) = \mathrm{cy}_k(i(b)) = b(x_k) = a\,.$$

It remains to check associativity. Let $t \in \mathbb{T}\mathbb{T}(\mathrm{TA}(S))$ and set $t' := \mathbb{T}\pi(t)$. For every vertex $v \in \mathrm{dom}(t)$, we fix a word u_v and a function σ_v such that $\pi(t(v)) = \mathrm{cy}_{\sigma_v}(i(\pi(u_v)))$. Let β be the branch of t' corresponding to $\mathrm{un}(t')$ and fix $\hat{u}$ and $\hat{\sigma}$ such that $\pi(t') = \mathrm{cy}_{\hat{\sigma}}(i(\pi(\hat{u})))$. Finally, fix u^* and σ^* such that $\pi(\mathrm{flat}(t)) = \mathrm{cy}_{\sigma^*}(i(\pi(u^*)))$. It follows that u^* consists of the concatenation of the u_v, for v on (a prefix of) β. Furthermore, each element of $\hat{u}$ corresponds to the product $\pi(u_v)$ for a suitable vertex v. Since the product of an ω-semigroup is associative it therefore follows that $\pi(u^*) = \pi(\hat{u})$. This implies that

$$\pi(\mathrm{flat}(t)) = \mathrm{cy}_{\sigma^*}(i(\pi(u^*))) = \mathrm{cy}_{\hat{\sigma}}(i(\pi(\hat{u}))) = \pi(\mathbb{T}\pi(t))\,.$$

(b) $(\Rightarrow)$ Let $t \in \mathbb{T}(\mathrm{TA}(S))$ and $T \in \mathbb{T}\mathbb{U}(\mathrm{TA}(S))$ be trees such that $t = \mathbb{T}\inf(T)$. Since every infimum $\inf T(v)$ is defined, it follows that either

- $t(v) \in S_{\mathrm{o}}$ and $T(v) \subseteq S_{\mathrm{o}}$, or

- $t(v) = a(x_k)$ and $T(v) = \{\, b(x_k) \mid b \in P_v \,\}$ for some set $P_v \subseteq S_1$.

Consequently, the unravellings of t and every $s \in^{\mathbb{T}} T$ have the same shape and correspond to the same path in t. This implies that

$$\mathrm{un}(t) = \mathbb{T}\inf(\mathrm{un}(T)).$$

Let $u \in \mathbb{W}S$ and $U \in \mathbb{W}\mathbb{U}(S)$ be the words corresponding to these two unravellings. Then $u = \mathbb{T}\inf U$ and meet-continuity of $\mathfrak{S}$ implies that

$$\pi(u) = \inf \{\, \pi(w) \mid w \in^{\mathbb{W}} U \,\}.$$

Consequently,

$$\pi(t) = \inf \{\, \pi(s) \mid s \in^{\mathbb{W}} T \,\}.$$

($\Leftarrow$) Let $u \in \mathbb{W}S$ and $U \in \mathbb{W}\mathbb{U}S$ be words with $u = \mathbb{W}\inf(U)$. We set $t := (\rho \circ \mathbb{W}i)(u)$ and $T := \mathbb{U}(\rho \circ \mathbb{W}i)(U)$. Then $\pi(t) = i(\pi(u))$ and $\pi(T) = \mathbb{U}i(\pi(U))$. As $\mathrm{TA}(S)$ is meet-continuous, we furthermore have

$$\pi(t) = \inf \pi(T).$$

Applying i to this equation it follows that $\pi(u) = \inf \pi(U)$. $\qquad\square$

Let us show that the functors $\mathrm{TA} : \mathsf{SGrp} \to \mathsf{Alg}$ and $\mathrm{SG} : \mathsf{Alg} \to \mathsf{SGrp}$ form an adjunction $\mathrm{TA} \vdash \mathrm{SG}$.

Proposition 4.20. *Let $\mathfrak{S}$ be an ω-semigroup and $\mathfrak{A}$ a tree algebra.*

(a) *For every morphism $\varphi : \mathfrak{S} \to \mathrm{SG}(\mathfrak{A})$ of ω-semigroups, there exists a unique morphism $\hat{\varphi} : \mathrm{TA}(\mathfrak{S}) \to \mathfrak{A}$ of tree algebras such that $\mathrm{SG}(\hat{\varphi}) = \varphi$.*

(b) *If φ is surjective, then $\mathrm{rng}\,\hat{\varphi} = \langle A_0 \cup A_1 \rangle$.*

Proof. (a) Let $a \in \mathrm{TA}_n(\mathfrak{S}) = S_0 \cup (S_1 \times [n])$. We define $\hat{\varphi}(a) \in A_n$ by

$$\hat{\varphi}(a) := \begin{cases} \varphi(a) & \text{if } a \in S_0 , \\ (\varphi(b))(x_i) & \text{if } a = b(x_i) \in S_1 \times [n] . \end{cases}$$

Then $\mathrm{SG}(\hat{\varphi}) = \varphi$ and $\hat{\varphi}$ is clearly the only possible function with this property. Hence, it remains to prove that $\hat{\varphi}$ is a morphism of tree algebras.

Let $\mathfrak{T} := \mathrm{TA}(\mathfrak{S})$. We start by noting that

$$\pi(\mathbb{T}\varphi(u)) = \varphi(\pi(u)), \quad \text{for trees } u \in \mathbb{T}(T_0 \cup T_1).$$

For the general case, consider a tree $t \in \mathbb{T}T$ and set $t' := \mathbb{T}\varphi(t)$. Using Lemma 4.8, we can find a T-trace u of t such that $\pi(u) = \pi(t)$. Let $\mathfrak{B} \subseteq \mathfrak{A}$ be the subalgebra of $\mathfrak{A}$ generated by $A_0 \cup A_1$. Then $\mathfrak{B}$ is semigroup-like and rng $\varphi \subseteq B$. Hence, we can use Lemma 4.8 to find a B-trace v' of $\mathbb{T}\varphi(t)$ such that $\pi(v') = \pi(\mathbb{T}\varphi(t))$. Fix a tree $v \in \mathbb{T}(T_0 \cup T_1)$ such that $v' = \mathbb{T}\varphi(v)$. As $\mathbb{T}\varphi(u)$ is an B-trace of $\mathbb{T}\varphi(t)'$ and v is a T-trace of t, it follows by Lemma 4.10 that

$$
\begin{aligned}
\varphi(\pi(t)) = \varphi(\pi(u)) = \pi(\mathbb{T}\varphi(u)) \\
\geq \pi(\mathbb{T}\varphi(t)) = \pi(v') = \pi(\mathbb{T}\varphi(v)) = \varphi(\pi(v)) \\
\geq \varphi(\pi(t)) \,.
\end{aligned}
$$

Consequently, $\varphi(\pi(t)) = \pi(\mathbb{T}\varphi(t))$.

(b) Let $\mathfrak{T} := \mathrm{TA}(\mathfrak{S})$. If φ is surjective, then $\hat{\varphi}[T_0] = A_0$ and $\hat{\varphi}[T_1] = A_1$. Hence,

$$
\varphi[\langle T_0 \cup T_1 \rangle] = \langle \varphi[T_0] \cup \varphi[T_1] \rangle = \langle A_0 \cup A_1 \rangle \,. \qquad \square
$$

As an application, let us prove the following characterisation of semigroup-like tree algebras.

Proposition 4.21. *A tree algebra $\mathfrak{A}$ is semigroup-like if, and only if, there exists a surjective morphism $\varphi : \mathrm{TA}(\mathfrak{S}) \to \mathfrak{A}$, for some ω-semigroup $\mathfrak{S}$.*

Proof. ($\Rightarrow$) Applying Proposition 4.20 to the identity morphism $\psi : \mathrm{SG}(\mathfrak{A}) \to \mathrm{SG}(\mathfrak{A})$, we obtain a morphism $\hat{\psi} : \mathrm{TA}(\mathrm{SG}(\mathfrak{A})) \to \mathfrak{A}$ with rng $\hat{\psi} = \langle A_0 \cup A_1 \rangle = A$. Hence, $\hat{\psi}$ is surjective.

($\Leftarrow$) Suppose that $\varphi : \mathrm{TA}(\mathfrak{S}) \to \mathfrak{A}$ is surjective and let T be the universe of $\mathrm{TA}(\mathfrak{S})$. Then

$$
\langle A_0 \cup A_1 \rangle = \langle \varphi[T_0 \cup T_1] \rangle = \varphi[\langle T_0 \cup T_1 \rangle] = \varphi[T] = A \,.
$$

Hence, $\mathfrak{A}$ is semigroup-like. $\qquad \square$

4.3 Skeletons and branch-continuity

After these preparations we are finally able to define the class of tree algebras we are interested in.

Definition 4.22. Let $\mathfrak{A}$ be a tree algebra.

(a) A semigroup-like subalgebra $\mathfrak{S} \subseteq \mathfrak{A}$ is a *skeleton* of $\mathfrak{A}$ if

- ◆ S is meet-continuously embedded in $\mathfrak{A}$ and

- ◆ $\mathrm{cl}(S)$ is a set of join-generators of $\mathfrak{A}$.

(b) A tree algebra $\mathfrak{A}$ is *branch-continuous* if $\mathfrak{A} \in \mathsf{CAlg}$ and it has a skeleton.

(c) We denote be BAlg the subcategory of CAlg consisting of all branch-continuous tree algebras and all morphisms that preserve meets and joins. ⌋

Let us start our investigation of branch-continuous tree algebras with a summary of how to compute products in them.

Lemma 4.23. *Let $\mathfrak{A}$ be a branch-continuous tree algebra, $\mathfrak{S} \subseteq \mathfrak{A}$ a skeleton of $\mathfrak{A}$, and $C := \mathrm{cl}(S)$.*

(a) $\pi(t) = \sup \big\{ \pi(s) \,\big|\, s \in \mathbb{T}C,\ s \leq^{\mathbb{T}} t \big\}, \qquad$ *for $t \in \mathbb{T}A$,*

(b) $\pi(t) = \inf \big\{ \pi(u) \,\big|\, u \text{ an } S\text{-trace of } t \big\}, \qquad$ *for $t \in \mathbb{T}C$.*

Proof. (a) follows by Lemma 2.19 (c); and (b) by Proposition 4.11. □

Branch-continuity is preserved by certain morphisms.

Lemma 4.24. *Let $\mathfrak{A}$ be a branch-continuous tree algebra and $\varphi : \mathfrak{A} \to \mathfrak{B}$ a surjective morphism that preserves meets and joins.*

(a) *If $\mathfrak{S}$ is a skeleton of $\mathfrak{A}$, then $\varphi[\mathfrak{S}]$ is one of $\mathfrak{B}$.*

(b) *$\mathfrak{B}$ is branch-continuous.*

Proof. (a) The image $T := \varphi[S]$ is a semigroup-like subalgebra of $\mathfrak{B}$ that, according to Lemma 2.17, is meet-continuously embedded in $\mathfrak{B}$. Since φ is surjective and it preserves meets and joins, it further follows that $\mathrm{cl}(T) = \varphi[\mathrm{cl}(S)]$ and that this is a set of join-generators of $\mathfrak{B}$.

(b) According to Lemma 2.16, $\mathfrak{B}$ is complete, distributive, and join-continuous. Furthermore, we have seen in (a) that $\mathfrak{B}$ has a skeleton. □

There are certain canonical branch-continuous tree algebras that are 'freely' generated by their skeleton. We will show below that every branch-continuous tree algebra is a quotient of an algebra of this form.

Definition 4.25. Let $\mathfrak{S}$ be an ω-semigroup. The *branch algebra* of $\mathfrak{S}$ is

$$\mathrm{Branch}(\mathfrak{S}) := \mathbb{DU}(\mathrm{TA}(\mathfrak{S})).$$ ⌋

Our first aim is to show that $\mathrm{Branch}(\mathfrak{S})$ is branch-continuous.

Theorem 4.26. *The branch algebra* $\mathrm{Branch}(\mathfrak{S})$ *associated with a meet-continuous* ω*-semigroup* $\mathfrak{S}$ *is branch-continuous.*

Proof. The fact that $\mathrm{Branch}(\mathfrak{S}) \in \mathrm{CAlg}$ follows by Propositions 4.19 and 3.8. Hence, it remains to prove that it has a skeleton.

Let T be the image of the canonical embedding $\mathrm{TA}(\mathfrak{S}) \to \mathrm{Branch}(\mathfrak{S})$ and set $C := \mathrm{cl}(T)$. We claim that T is a skeleton of $\mathrm{Branch}(\mathfrak{S})$. First, note that $\mathbb{U}T = \mathrm{cl}(T) = C$ since $\mathbb{U}T$ is meet-generated by T and closed under meets. Furthermore, $\mathbb{D}\mathbb{U}T = \mathrm{Branch}(S)$ is the closure of C under joins. Hence, C is a set of join generators of $\mathrm{Branch}(\mathfrak{S})$.

To conclude the proof, it remains to prove that T is meet-continuously embedded in $\mathrm{Branch}(\mathfrak{S})$. We have seen in Propositions 3.11 and 3.19 that the embedding $\eta : \mathbb{U}\mathrm{TA}(\mathfrak{S}) \to \mathbb{D}\mathbb{U}\mathrm{TA}(\mathfrak{S})$ preserves meets and that the algebra $\mathbb{U}(\mathrm{TA}(\mathfrak{S}))$ is meet-continuous. Hence, the image C of η is meet-continuously embedded in $\mathrm{Branch}(\mathfrak{S})$. In particular, so is $T \subseteq C$. □

Proposition 4.27. *Let* $\mathfrak{A}$ *be a branch-continuous tree algebra with skeleton* $\mathfrak{S}$ *and let* $\mathfrak{U}$ *be a meet-continuous* ω*-semigroup.*

(a) *For every morphism* $\varphi : \mathfrak{U} \to \mathrm{SG}(\mathfrak{S})$ *of* ω*-semigroups, there exists a unique morphism* $\hat{\varphi} : \mathrm{Branch}(\mathfrak{U}) \to \mathfrak{A}$ *of tree algebras such that* $\mathrm{SG}(\hat{\varphi})$ *extends* φ *and* $\hat{\varphi}$ *preserves arbitrary joins and meets.*

(b) *If* φ *is surjective, so is* $\hat{\varphi}$.

Proof. (a) Let $\mathfrak{T} := \mathrm{TA}(\mathfrak{U})$. By Proposition 4.20, there exists a unique morphism $\varphi_0 : \mathfrak{T} \to \mathfrak{S}$ extending φ. By Proposition 3.22, we can extend φ_0 to a unique meet-preserving morphism $\varphi_1 : \mathbb{U}\mathfrak{T} \to \mathfrak{A}$ by setting

$$\varphi_1(J) := \inf \varphi_0[J], \quad \text{for } J \in \mathbb{U}\mathfrak{T}.$$

Finally, we use Proposition 3.12 to extend φ_1 to a unique join-preserving morphism $\hat{\varphi} : \mathbb{D}\mathbb{U}\mathfrak{T} \to \mathfrak{A}$ by setting

$$\hat{\varphi}(I) := \sup \varphi_1[I], \quad \text{for } I \in \mathbb{D}\mathbb{U}\mathfrak{T}.$$

Note that Lemma 2.20 implies that $\hat{\varphi}$ preserves arbitrary meets.

(b) If φ is surjective, so is the morphism $\varphi_0 : \mathrm{TA}(\mathfrak{U}) \to \mathfrak{S}$ from the proof of (a), i.e., $\mathrm{rng}\, \varphi_0 = S$. As φ_1 preserves arbitrary meets, its range includes the closure $C := \mathrm{cl}(S)$. Similarly, the range of $\hat{\varphi}$ includes the closure of C under joins, which is all of A. Thus, $\hat{\varphi}$ is surjective. □

As promised above, we can show that, conversely, every branch-continuous tree algebra is a quotient of an algebra of the form $\mathrm{Branch}(\mathfrak{S})$.

Theorem 4.28. *Let $\mathfrak{A}$ be a branch-continuous tree algebra with skeleton $\mathfrak{S}$. There exists a surjective morphism $\varphi : \mathrm{Branch}(\mathrm{SG}(\mathfrak{S})) \to \mathfrak{A}$ that preserves joins and meets.*

Proof. Let $\psi : \mathrm{SG}(\mathfrak{S}) \to \mathrm{SG}(\mathfrak{S})$ be the identity morphism. By Proposition 4.27, there exists a unique extension $\hat{\psi} : \mathrm{Branch}(\mathrm{SG}(\mathfrak{S})) \to \mathfrak{A}$ which preserves joins and meets. □

Combining this theorem with Lemma 4.24, we obtain the following characterisation of branch-continuous tree algebras as quotients of an algebra of the form $\mathrm{Branch}(\mathfrak{S})$.

Corollary 4.29. *A tree algebra $\mathfrak{A}$ is branch-continuous if, and only if, there exists an ω-semigroup $\mathfrak{S}$ and a surjective morphism $\varphi : \mathrm{Branch}(\mathfrak{S}) \to \mathfrak{A}$ that preserves joins and meets.*

4.4 Closure under products

Our next goal is to prove that the class of branch-continuous tree algebras is closed under finite products.

Definition 4.30. Let $\mathfrak{A}^i = \langle A^i, \pi^i, \leq^i \rangle$, for $i \in I$, be a family of tree algebras. The *product* $\prod_{i \in I} \mathfrak{A}^i$ is the tree algebra with domains

$$\prod_{i \in I} A_n^i\,, \quad \text{for } n < \omega\,,$$

order

$$(a_i)_{i \in I} \leq (b_i)_{i \in I} \quad :\text{iff} \quad a_i \leq^i b_i\,, \quad \text{for all } i \in I\,,$$

and product

$$\pi(t) := \big(\pi^i(\mathbb{T}p_i(t))\big)_{i \in I}$$

where the function $p_k : \prod_{i \in I} A^i \to A^k$ projects a sequence $(a_i)_{i \in I}$ to its k-th component a_k.

Lemma 4.31. *Let $\mathfrak{A}^i = \langle A^i, \pi^i, \leq^i \rangle$, for $i \in I$, be a family of tree algebras.*

(a) $\prod_{i\in I}\mathfrak{A}^i$ is a tree algebra.

(b) Each projection $p_k : \prod_{i\in I}\mathfrak{A}^i \to \mathfrak{A}^k$ is a morphism of tree algebras preserving arbitrary joins and meets.

(c) If every $\mathfrak{A}^i$ is complete, distributive, and join-continuous, then so is $\prod_{i\in I}\mathfrak{A}^i$.

Proof. (b) It follows immediately from the definitions that p_k is monotone, that it commutes with products, and that is preserves joins and meets.

(a) Clearly, the product π is monotone. Furthermore,

$$p_i(\pi(\mathrm{sing}(a))) = \pi^i(p_i(\mathrm{sing}(a)) = \pi^i(\mathrm{sing}(p_i(a))) = p_i(a)\,,$$

which implies that $\pi \circ \mathrm{sing} = \mathrm{id}$.

Hence, it remains to prove associativity. Let $t \in \mathbb{T}_n\mathbb{T}(\prod_i A^i)$. Then

$$\begin{aligned}
p_i(\pi(\mathrm{flat}(t))) &= \pi^i(\mathbb{T}p_i(\mathrm{flat}(t)))\\
&= \pi^i(\mathrm{flat}(\mathbb{T}p_i(t)))\\
&= \pi^i(\mathbb{T}\pi^i(\mathbb{T}p_i(t)))\\
&= \pi^i(\mathbb{T}p_i(\mathbb{T}\pi(t))) = p_i(\pi(\mathbb{T}\pi(t)))\,,
\end{aligned}$$

which implies that $\pi \circ \mathrm{flat} = \pi \circ \mathbb{T}\pi$.

(c) For completeness, let $X \subseteq \prod_i A^i$. As p_i commutes with joins, we have

$$p_i(\sup X) = \sup p_i[X]\,,$$

which implies that $\sup X = (\sup p_i[X])_{i\in I}$. Similarly, it follows that $\inf X = (\inf p_i[X])_{i\in I}$.

In the same way it follows that the product is distributive. For join-continuity, let $t = \mathbb{T}\sup(S)$. Then

$$\mathbb{T}p_i(t) = \mathbb{T}p_i(\mathbb{T}\sup(S)) = \mathbb{T}\sup(\mathbb{T}p_i(S))$$

which, by join-continuity of $\mathfrak{A}^i$, implies that

$$\pi^i(\mathbb{T}p_i(t)) = \sup\{\,\pi^i(s) \mid s \in^{\mathbb{T}} \mathbb{T}p_i(S)\,\}\,.$$

Consequently,

$$\begin{aligned}
p_i(\pi(t)) &= \pi^i(\mathbb{T}p_i(t))\\
&= \sup\{\,\pi^i(s) \mid s \in^{\mathbb{T}} \mathbb{T}p_i(S)\,\}\\
&= \sup\{\,\pi^i(\mathbb{T}p_i(s)) \mid s \in^{\mathbb{T}} S\,\}\\
&= \sup\{\,p_i(\pi(s)) \mid s \in^{\mathbb{T}} S\,\} = p_i(\sup\{\,\pi(s) \mid s \in^{\mathbb{T}} S\,\})\,. \qquad \square
\end{aligned}$$

Proposition 4.32. *Let $\varphi^i : \mathfrak{B} \to \mathfrak{A}^i$, $i \in I$, be a family of tree algebra morphisms. There exists a unique morphism $\psi : \mathfrak{B} \to \prod_{i \in I} \mathfrak{A}^i$ such that*

$$\varphi^i = p_i \circ \psi, \quad \text{for all } i \in I.$$

Proof. The function $\psi(b) := (\varphi^i(b))_{i \in I}$ has the desired properties. $\square$

Theorem 4.33. *If $\mathfrak{A}$ and $\mathfrak{B}$ are branch-continuous tree algebras, so is their product $\mathfrak{A} \times \mathfrak{B}$.*

Proof. We have seen in Lemma 4.31, that $\mathfrak{A} \times \mathfrak{B}$ is complete, distributive, and join-continuous. Hence, it remains to find a skeleton of $\mathfrak{A} \times \mathfrak{B}$. Let $\mathfrak{S}$ and $\mathfrak{T}$ be skeletons of, respectively, $\mathfrak{A}$ and $\mathfrak{B}$. We claim that $\mathfrak{S} \times \mathfrak{T}$ is a one of the product.

Note that we have shown in Lemma 4.31 that the projections $p_0 : A \times B \to a$ and $p_1 : A \times B \to B$ preserve projections and arbitrary meets and joins. Consequently, we have $\mathrm{cl}(S \times T) = \mathrm{cl}(S) \times \mathrm{cl}(T)$. Furthermore, the fact that $\mathrm{cl}(S)$ and $\mathrm{cl}(T)$ are sets of join-generators implies that so is $\mathrm{cl}(S) \times \mathrm{cl}(T)$.

It remains to show that $S \times T$ is meet-continuously embedded in $\mathfrak{A} \times \mathfrak{B}$. Let $t \in \mathbb{T}(A \times B)$ and $U \in \mathbb{TU}(S \times T)$ be trees with $t = \mathbb{T}\inf(U)$. Applying the projection p_i, we obtain

$$\pi(\mathbb{T}p_i(t)) = p_i(\pi(t)) = p_i(\pi(\mathbb{T}\inf U)) = \pi(\mathbb{T}\inf(\mathbb{T}p_i(U))),$$

which implies that

$$\pi(\mathbb{T}p_i(t)) = \inf \left\{ \pi(s) \mid s \in^{\mathbb{T}} \mathbb{T}p_i(U) \right\}.$$

It follows that $\pi(t) = \inf \left\{ \pi(s) \mid s \in^{\mathbb{T}} U \right\}$. $\square$

4.5 Regular languages and recognisability

Having introduced branch-continuous tree algebras we can use them to give a characterisation of the class of regular languages. We will use the ordered version of recognisability.

Definition 4.34. Let $\mathfrak{A}$ be a tree algebra. A subset $L \subseteq A_m$ is *recognised* by a morphism $\eta : \mathfrak{A} \to \mathfrak{B}$ if $L = \eta^{-1}[P]$ for some upwards closed subset $P \subseteq B_m$. We say that L is *recognised* by $\mathfrak{B}$ if it is recognised by some morphism $\mathfrak{A} \to \mathfrak{B}$. ⌋

We will show that a tree language is recognisable if, and only if, it is recognised by a finitary, branch-continuous tree algebra. We start by showing that recognisable languages are regular. When taking a closer look at what it means for a recognisable language to be regular, we arrive at the following definition, which has recently been introduced in [4].

Definition 4.35. A tree algebra $\mathfrak{A}$ is *regular* if it is finitary and there exists a finite set $C \subseteq A$ of generators such that, for every element $a \in A$, the preimage

$$\pi^{-1}(a) \cap \mathbb{T}C \text{ is a regular language.} \qquad \lrcorner$$

It is straightforward to check that the regular tree algebras recognise precisely the regular tree languages.

Theorem 4.36 ([4]). *A tree algebra $\mathfrak{A}$ is regular if, and only if, every language recognised by $\mathfrak{A}$ is regular.*

Of course, the definition of a regular algebra was specifically chosen to make this theorem true. But because of its cyclic nature it does not further our understanding of the regular tree languages. What is missing is a good algebraic characterisation telling us how regular algebras look like. Branch-continuous algebras *do* have such a characterisation and can therefore serve as an alternative approach to regularity. We start by showing that every branch-continuous algebra is regular.

Proposition 4.37. *Every finitary, branch-continuous tree algebra is regular.*

Proof. Let $\mathfrak{A}$ be finitary and branch-continuous, $\mathfrak{S}$ a skeleton of $\mathfrak{A}$, and set $C :=$ cl(S). Fix a finite set $B \subseteq A$ of generators. W.l.o.g. we may assume that $B = A_0 \cup \cdots \cup A_{k-1}$, for some $k < \omega$. We will construct MSO-formulae φ_a defining the languages

$$\pi^{-1}(a) \cap \mathbb{T}B, \quad \text{for } a \in A.$$

First, note that, given a tree $t \in \mathbb{T}_n B$, we can encode an S-trace u of t by a family $(U_c)_{c \in S}$ of unary predicates such that the union $\bigcup_c U_c$ contains the branch corresponding to the S-trace u and the various predicates U_c encode its labelling. Since, in monadic second-order logic, we can compute infinite products in finite ω-semigroups, there are formulae $\vartheta_a(\bar{Z})$, for $a \in A$, that check whether $\pi(u) = a$ when given a tree $t \in \mathbb{T}_n(B \cap C)$ and an S-trace u of t that is encoded in $\bar{Z}$.

For trees $t \in \mathbb{T}_n C$, we have seen in Lemma 4.23 (b) that

$$\pi(t) = \inf \left\{ \pi(u) \mid u \text{ an } S\text{-trace of } t \right\}.$$

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Consequently, can use the formulae $\vartheta_a(\bar{Z})$ to construct formulae ψ_a that, given a tree $t \in \mathbb{T}(B \cap C)$, check whether $\pi(t) = a$.

Finally, according to Lemma 4.23 (a), we have

$$\pi(t) = \sup\left\{\, \pi(s) \;\middle|\; s \in \mathbb{T}_n C,\ s \leq^{\mathbb{T}} t \,\right\}, \quad \text{for all trees } t\,.$$

Therefore, we can use the above formulae ψ_a to construct formulae φ_a, for $a \in A$, checking whether the product of a given tree $t \in \mathbb{T}_n B$ evaluates to a. $\qquad \square$

It remains to prove the converse: given a regular language we have to find a branch-continuous algebra recognising it. We start by fixing our terminology regarding automata.

Definition 4.38. Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega\rangle$ be a nondeterministic parity automaton and set $D := \operatorname{rng}\Omega$.

(a) Let $t \in \mathbb{T}_n\Sigma$. A *run* of $\mathcal{A}$ on a tree $t \in \mathbb{T}_n\Sigma$ is a tree $\rho \in \mathbb{T}_0 Q$ with the same domain as t that satisfies the following two conditions:

- for every vertex $v \in \operatorname{dom}(t) \smallsetminus \operatorname{Hole}(t)$ with $\operatorname{ar}(t(v)) = n$ and immediate successors $u_0, \ldots, u_{n-1}$,

$$\langle \rho(v), t(v), \rho(u_0), \ldots, \rho(u_{n-1})\rangle \in \Delta_n\,;$$

- for every infinite branch β of t,

$$\liminf_{v < \beta} \Omega(\rho(v)) \quad \text{is even.}$$

(b) The *profile* of a run ρ on a tree $t \in \mathbb{T}_n\Sigma$ is the pair

$$\operatorname{pf}(\rho) := \langle \rho(\langle\rangle), \bar{u}\rangle,$$

where

$$u_i := \begin{cases} \langle d, \rho(v_i)\rangle & \text{if } v_i := \operatorname{hole}_i(t) \text{ is defined and} \\ & \qquad d := \min\left\{\, \Omega(\rho(z)) \mid z \leq v_i \,\right\}, \\ \bot & \text{otherwise}\,. \end{cases}$$

Definition 4.39. Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, \Omega\rangle$ be a nondeterministic parity automaton and set $D := \operatorname{rng}\Omega$.

(a) The *automaton ω-semigroup* $\mathfrak{S}_{\mathcal{A}}$ associated with $\mathcal{A}$ is the partial ω-semigroup with domains

$$S_0 := Q \quad \text{and} \quad S_1 := Q \times D \times Q \,.$$

The order is equality on S_0 and on S_1 it is given by

$$\langle p, k, q \rangle \leq \langle p', k', q' \rangle \quad :\text{iff} \quad p = p', \ q = q', \text{ and } k \sqsubseteq k' \text{ in the ordering}$$
$$1 \sqsubset 3 \sqsubset 5 \sqsubset \cdots \sqsubset 4 \sqsubset 2 \sqsubset 0 \,.$$

(The closer a priority is to acceptance, the larger it is.) The product is determined by the equations

$$\langle p, k, q \rangle \cdot q' := \begin{cases} p & \text{if } q = q' \,, \\ \text{undefined} & \text{otherwise} \,, \end{cases}$$

$$\langle p, k, q \rangle \cdot \langle p', k', q' \rangle := \begin{cases} \langle p, l, q' \rangle & \text{if } q = p' \text{ and } l := \min\{k, k'\} \,, \\ \text{undefined} & \text{otherwise} \,, \end{cases}$$

$$\prod_{n < \omega} \langle p_n, k_n, q_n \rangle := \begin{cases} p_0 & \text{if } q_n = p_{n+1} \text{ for all } n \text{ and} \\ & \liminf_{n \to \infty} k_n \text{ is even}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(b) We define a morphism $\alpha_{\mathcal{A}} : \mathbb{T}\Sigma \to \mathrm{Branch}(\mathfrak{S}_{\mathcal{A}})$ as follows. Given a tree $t \in \mathbb{T}_n\Sigma$, we set

$$\alpha_{\mathcal{A}}(t) := \Downarrow \big\{\, \eta(\widetilde{\mathrm{pf}}(\rho)) \mid \rho \text{ a run on } t \,\big\} \,,$$

where $\eta : \mathbb{U}(\mathrm{TA}(\mathfrak{S}_{\mathcal{A}})) \to \mathrm{Branch}(\mathfrak{S}_{\mathcal{A}})$ is the canonical embedding and, for a run ρ with profile

$$\mathrm{pf}(\rho) = \langle p, u_0, \ldots, u_{n-1} \rangle \,,$$

we have set

$$\widetilde{\mathrm{pf}}(\rho) := \tilde{p} \sqcap \tilde{u}_0 \sqcap \cdots \sqcap \tilde{u}_{n-1} \in \mathbb{U}(\mathrm{TA}(S_{\mathcal{A}}))$$

with

$$\tilde{u}_i := \begin{cases} \langle p, k, q \rangle(x_i) & \text{if } u_i = \langle k, q \rangle \,, \\ \top & \text{if } u_i = \bot \,, \end{cases}$$

$$\tilde{p} := \begin{cases} p & \text{if } t \text{ has an infinite branch or a leaf that is not a hole,} \\ \top & \text{otherwise.} \end{cases}$$

Lemma 4.40. $\alpha_\mathcal{A} : \mathbb{T}\Sigma \to \mathrm{Branch}(\mathfrak{S}_\mathcal{A})$ *is a morphism of tree algebras recognising* $L(\mathcal{A})$.

Proof. To see that $\alpha_\mathcal{A}$ recognises $L(\mathcal{A})$, let

$$P := \left\{ I \subseteq \mathbb{U}(\mathrm{TA}(S_\mathcal{A})) \mid \eta(q_\circ) \in I \text{ and } I \text{ is upwards closed w.r.t. } \subseteq \right\}$$
$$\subseteq \mathrm{Branch}(S_\mathcal{A}).$$

For a tree $t \in \mathbb{T}_\circ \Sigma$, it follows that

$$
\begin{array}{lll}
t \in L(\mathcal{A}) & \text{iff} & \text{there is a run } \rho \text{ of } \mathcal{A} \text{ on } t \text{ such that } \rho(\langle\rangle) = q_\circ \\
& \text{iff} & \text{there is a run } \rho \text{ of } \mathcal{A} \text{ on } t \text{ such that } \widetilde{\mathrm{pf}}(\rho) = q_\circ \\
& \text{iff} & \eta(q_\circ) \in \alpha_\mathcal{A}(t) \\
& \text{iff} & \alpha_\mathcal{A}(t) \in P.
\end{array}
$$

It remains to check that $\alpha_\mathcal{A}$ is a morphism. For a tree $t \in \mathbb{T}_n \mathbb{T}\Sigma$, we have

$$
\begin{aligned}
\pi(\mathbb{T}\alpha_\mathcal{A}(t)) &= \left\{ a \mid a \le \pi(s),\ s \in^{\mathbb{T}} \mathbb{T}\alpha_\mathcal{A}(t),\ \pi(s) \text{ defined} \right\} \\
&= \left\{ a \mid a \le \pi(s),\ s \simeq_{\mathrm{sh}} t,\ \pi(s) \text{ defined}, \right. \\
&\qquad\qquad \left. s(v) \in \alpha_\mathcal{A}(t(v)),\ \text{for all } v \right\} \\
&= \left\{ a \mid a \le \pi(s),\ s \simeq_{\mathrm{sh}} t,\ \pi(s) \text{ defined, for each } v \text{ there is} \right. \\
&\qquad\qquad \left. \text{a run } \rho_v \text{ on } t(v) \text{ such that } s(v) = \eta(\widetilde{\mathrm{pf}}(\rho_v)) \right\}.
\end{aligned}
$$

We have to show that this set is equal to

$$\alpha_\mathcal{A}(\mathrm{flat}(t)) = \left\{ a \mid a \le \eta(\widetilde{\mathrm{pf}}(\rho)) \text{ for some run } \rho \text{ on } \mathrm{flat}(t) \right\}.$$

($\supseteq$) Let ρ be a run on $\mathrm{flat}(t)$. For $v \in \mathrm{dom}(t)$, let ρ_v be the restriction of ρ to the vertices in $\mathrm{dom}(t(v))$ and set $s(v) := \eta(\widetilde{\mathrm{pf}}(\rho_v))$. Then $\eta(\widetilde{\mathrm{pf}}(\rho)) = \pi(s)$.

($\subseteq$) Let s be a tree with $s(v) = \eta(\widetilde{\mathrm{pf}}(\rho_v))$, for some run ρ_v on $t(v)$. Let ρ be the run on $\mathrm{flat}(t)$ such that, for every $v \in \mathrm{dom}(t)$, the restriction of ρ to the vertices in $\mathrm{dom}(t(v))$ coincides with ρ_v. Then $\eta(\widetilde{\mathrm{pf}}(\rho)) = \pi(s)$. $\qquad\square$

Lemma 4.41. *The automaton ω-semigroup $\mathfrak{S}_\mathcal{A}$ is meet-continuous.*

Proof. Let $U \in \mathbb{WP}(S)$. We have to show that

$$\pi(\mathbb{W}\inf(U)) = \inf \left\{ \pi(u) \mid u \in^{\mathbb{W}} U \right\}.$$

We distinguish several cases.

(1) First, suppose that $\mathbb{W}\inf(U)$ is undefined. Then $\inf U(i)$ is undefined, for some index i. This means that $U(i)$ contains two incomparable elements. Fix sequences $u, u' \in^{\mathbb{W}} U$ such that $u(i)$ and $u'(i)$ are incomparable and $u(j) = u'(j)$, for all $j \neq i$. If at least one of $\pi(u)$ and $\pi(u')$ is not defined, we are done. Hence, suppose that both products are defined. We claim that their values are incomparable and, thus, the infimum on the right-hand side of the above equation is not defined. For the proof, we distinguish several cases.

(1 a) Suppose that $U(i)$ has arity 0. Then i is the last position. Let $u(i) = p$ and $u'(i) = p'$. If $i = 0$, then $\pi(u) = p \neq p' = \pi(u')$ are incomparable. Hence, suppose that $i > 0$ and let $u(i-1) = \langle q, k, r \rangle$. By assumption, $\pi(u)$ and $\pi(u')$ are both defined. This implies that $p = r = p'$. A contradiction.

(1 b) Suppose that $U(i)$ has arity 1. Let $u(i) = \langle p, k, q \rangle$ and $u'(i) = \langle p', k', q' \rangle$. Since these values are incomparable, we have $p \neq p'$ or $q \neq q'$.

First, suppose that $p \neq p'$. If $i > 0$ we can use the value of $u(i-1)$ to show that $p = p'$ as in Case (1 a) above. A contradiction. Consequently, $i = 0$ and, depending on the arity, we have either $\pi(u) = p \neq p' = \pi(u')$ or

$$\pi(u) = \langle p, k, r \rangle \neq \langle p', k', r' \rangle = \pi(u'),$$

for suitable $r, r' \in Q$ and $k, k' < \omega$.

Similarly, suppose that $q \neq q'$. Again, if i is not the last position, we get a contradiction by considering the value $u(i+1)$. It follows that

$$\pi(u) = \langle r, k, q \rangle \neq \langle r', k', q' \rangle = \pi(u'),$$

for suitable $r, r' \in Q$ and $k, k' < \omega$.

(2) It remains to consider the case where $\mathbb{W}\inf(U)$ is defined. For every position i in the sequence U, it follows that either

$$U(i) \subseteq \{p_i\}, \qquad\qquad \text{for some state } p_i,$$
$$\text{or} \quad U(i) \subseteq \{p_i\} \times K_i \times \{q_i\}, \qquad \text{for } p_i, q_i \in Q \text{ and } K_i \subseteq \omega.$$

Hence,

$$\inf U(i) = \langle p_i, k_i, q_i \rangle,$$

where $k_i := \inf_{\sqsubseteq} K_i$ is the $\sqsubseteq$-least element of K_i. We again distinguish several cases.

(2 a) Suppose that $q_i \neq p_{i+1}$, for some i. Then $\pi(u)$ is undefined, for all $u \in^{\mathbb{W}} U$, and so is $\pi(\mathbb{W}\inf(U))$. Hence, both sides of the equation are undefined.

(2 b) Suppose that $q_i = p_{i+1}$, for all i, and the sequence U is infinite. For every $u \in^{\mathbb{W}} U$, we have $u(i) = \langle p_i, m_i, q_i \rangle$, for some $m_i \in K_i$. Consequently, $\liminf_i k_i \sqsubseteq \liminf_i m_i$.

If $\liminf_i k_i$ is even, so is $\liminf_i m_i$. This implies that all products $\pi(u)$ are defined and so is $\pi(\mathbb{W}\inf(U))$. Consequently, $\pi(u) = p_0 = \pi(\mathbb{W}\inf(U))$.

If $\liminf_i k_i$ is odd, $\pi(\mathbb{W}\inf(U))$ is undefined. Choosing $u \in^{\mathbb{W}} U$ with $u(i) = \langle p_i, k_i, q_i \rangle$, it follows that $\pi(u)$ and, therefore, the infimum on the right-hand side of the equation is also undefined.

(2 c) Suppose that $q_i = p_{i+1}$, for all i, and the sequence U has length $n < \omega$. If the last element of U has arity 0, then $\pi(\mathbb{W}\inf(U)) = p_0$ and $\pi(u) = p_0$, for all $u \in^{\mathbb{W}} U$. Otherwise, we have $\pi(\mathbb{W}\inf(U)) = \langle p_0, \inf_i k_i, q_{n-1} \rangle$ and, for $u \in^{\mathbb{W}} U$ with $u(i) = \langle p_i, m_i, q_i \rangle$, $\pi(u) = \langle p_0, \inf_i m_i, q_{n-1} \rangle$ where $k_i \sqsubseteq m_i$. As above, we can chose $u \in^{\mathbb{W}} U$ with $m_i = k_i$. Consequently, the infimum on the right-hand side also evaluates to $\langle p_0, \inf_i k_i, q_{n-1} \rangle$. $\qquad\square$

Theorem 4.42. *Let Σ be a finite alphabet and $L \subseteq \mathbb{T}_0\Sigma$. The following statements are equivalent.*

(1) *L is* MSO-*definable.*

(2) *L is recognised by some nondeterministic parity automaton.*

(3) *L is recognised by a morphism $\varphi : \mathbb{T}\Sigma \to \text{Branch}(\mathfrak{S})$ for some finite, meet-continuous ω-semigroup $\mathfrak{S}$.*

(4) *L is recognised by some morphism $\varphi : \mathbb{T}\Sigma \to \mathfrak{A}$ to a finitary, branch-continuous tree algebra $\mathfrak{A}$.*

Proof. (1) $\Leftrightarrow$ (2) is standard; the implication (4) $\Rightarrow$ (1) was proved in Proposition 4.37; and (3) $\Rightarrow$ (4) holds by Theorem 4.26. Finally, the implication (2) $\Rightarrow$ (3) follows by Lemmas 4.40 and 4.41. $\qquad\square$

5 RT-algebras

5.1 Regular trees and unravellings

When we want to compute tree algebras we have to represent them in a finite way. Even for a finitary algebra, two problems arise: there are infinitely many sorts and the product $\pi : \mathbb{T}A \to A$ has an infinite domain. In this section, we look at finite

representations of the product function. We start by looking at algebras where the product is only defined for *regular* trees. Such algebras correspond to Wilke algebras in the semigroup setting. For lack of a better name, we will call them *RT-algebras*. (The term 'regular tree algebra' is unfortunately already taken.)

Definition 5.1. (a) We denote by $\mathbb{T}_n^{\mathrm{reg}} A$ the subset of $\mathbb{T}_n A$ consisting of all *regular* trees.

(b) $\mathfrak{A} = \langle A, \pi, \leq \rangle$ is an *RT-algebra* if $\pi : \mathbb{T}^{\mathrm{reg}} A \to A$ is a $\mathbb{T}^{\mathrm{reg}}$-algebra.

(c) The *regular restriction* of a tree algebra $\mathfrak{A} = \langle A, \pi, \leq \rangle$ is the corresponding RT-algebra

$$\mathfrak{A}^{\mathrm{reg}} := \langle A,\ \pi \restriction \mathbb{T}^{\mathrm{reg}} A,\ \leq \rangle .$$ ⌟

Note that RT-algebras can be seen as a particular form of partial tree algebras. Hence, many definitions and theorems about tree algebras apply. Furthermore, properties of tree algebras that are defined solely in terms of the order and finite products directly transfer from a tree algebra to the corresponding RT-algebra. For examples, the algebras $\mathfrak{A}$ and $\mathfrak{A}^{\mathrm{reg}}$ have the same cylinder maps and the same sets of join-generators.

One way to define regular trees is as unravellings of finite graphs. As we are dealing with trees where the successors are ordered from left-to-right, we need to do the same in our graphs. For this reason we label the edges by natural numbers to distinguish the successors of a vertex.

Definition 5.2. Let A be a ranked set.

(a) An *A-labelled graph* $\mathfrak{G} = \langle V, E, \lambda, \eta, v_{\mathrm{o}} \rangle$ consists of a directed graph $\langle V, E \rangle$ with a distinguished *root vertex* $v_{\mathrm{o}} \in V$ and two labelling functions $\lambda : V \to A$ and $\eta : E \to \omega$ such that every vertex $v \in V$ has exactly $n := \mathrm{ar}(\lambda(v))$ outgoing edges $e_{\mathrm{o}}, \dots, e_{n-1}$ and their labels are $\eta(e_i) = i$, for $i < n$. We call the end vertex of e_i the *i-th successor* of v.

(b) The *unravelling* $\mathrm{un}(\mathfrak{G})$ of an *A*-labelled graph $\mathfrak{G} = \langle V, E, \lambda, \eta, v_{\mathrm{o}} \rangle$ is the *A*-labelled tree whose vertices are all paths of $\mathfrak{G}$ that start at the root v_{o} and each such path is labelled by the label in $\mathfrak{G}$ of its end vertex. For two graphs $\mathfrak{G}$ and $\mathfrak{H}$, we write $\mathfrak{G} \simeq_{\mathrm{un}} \mathfrak{H}$ if they have the same unravelling.

(c) We denote by $\mathbb{G}_n A$ the set of all finite graphs whose unravelling is a tree in $\mathbb{T}_n A$. Let $\mathrm{un}_A : \mathbb{G} A \to \mathbb{T} A$ be the function mapping each graph to its unravelling and let $\mathrm{flat}_A : \mathbb{G}\mathbb{G} A \to \mathbb{G} A$ be the flattening function for graphs (which is defined in the natural way).

(d) For $G, G' \in \mathbb{G}A$, we write $G \simeq_{\mathrm{sh}} G'$ if these graphs only differ in the vertex labelling with respect to A, i.e., they have the same sets of vertices and edges and the same vertices are labelled by variables x_i.

Remark. $\mathrm{un}(\mathrm{flat}(G)) = \mathrm{flat}(\mathrm{un}(\mathbb{G}\mathrm{un}(G)))$, for all $G \in \mathbb{G}\mathbb{G}A$.

$$
\begin{array}{ccccc}
\mathbb{G}\mathbb{G}A & \xrightarrow{\;\;\mathbb{G}\mathrm{un}\;\;} & \mathbb{G}\mathbb{T}A & \xrightarrow{\;\;\mathrm{un}\;\;} & \mathbb{T}\mathbb{T}A \\[2pt]
\Big\downarrow{\scriptstyle\mathrm{flat}} & & & & \Big\downarrow{\scriptstyle\mathrm{flat}} \\[6pt]
\mathbb{G}A & & \xrightarrow{\qquad\qquad\mathrm{un}\qquad\qquad} & & \mathbb{T}A
\end{array}
$$

For most regular trees are the unravelling of several graphs. The following technical results help us in choosing a convenient one.

Lemma 5.3. *For all $G \in \mathbb{G}_\circ A$ and $H \in \mathbb{G}_\circ B$, there exist graphs $G' \in \mathbb{G}_\circ A$ and $H' \in \mathbb{G}_\circ B$ such that*

$$ G \simeq_{\mathrm{un}} G' \simeq_{\mathrm{sh}} H' \simeq_{\mathrm{un}} H . $$

Proof. The direct product $K := G \times H$ is a finite $(A \times B)$-labelled graph. Let G' and H' be the graphs obtained from K by projecting the labels to their two components. Then $G \simeq_{\mathrm{un}} G'$ and $H \simeq_{\mathrm{un}} H'$. Furthermore, $G' \simeq_{\mathrm{sh}} H'$. $\qquad\square$

Corollary 5.4. *Let $t_\circ, \ldots, t_m \in \mathbb{T}_n^{\mathrm{reg}}A$ with $m, n < \omega$. If $t_\circ \simeq_{\mathrm{sh}} \cdots \simeq_{\mathrm{sh}} t_m$, there exist finite graphs $G_\circ \simeq_{\mathrm{sh}} \cdots \simeq_{\mathrm{sh}} G_m$ such that $t_i = \mathrm{un}(G_i)$, for all $i \le m$.*

Proof. As each tree t_i contains only finitely many variables, we can decompose it as $t_i = p_i(s_\circ^i, \ldots, s_l^i)$ where p_i is a finite tree and each s_k^i either is a tree without variables or $s_k^i = \mathrm{sing}(x_j)$, for some variable x_j. Since $t_\circ \simeq_{\mathrm{sh}} \cdots \simeq_{\mathrm{sh}} t_m$, we can choose these trees such that

$$ p_\circ \simeq_{\mathrm{sh}} \cdots \simeq_{\mathrm{sh}} p_m \quad \text{and} \quad s_k^\circ \simeq_{\mathrm{sh}} \cdots \simeq_{\mathrm{sh}} s_k^m, \quad \text{for all } k \le l . $$

For those k where s_k^i does not contain variables, we can use Lemma 5.3 to find finite graphs

$$ H_k^\circ \simeq_{\mathrm{sh}} \cdots \simeq_{\mathrm{sh}} H_k^m \quad \text{with} \quad \mathrm{un}(H_k^i) = s_k^i . $$

For indices k with $s_k^i = \mathrm{sing}(x_j)$, we choose for H_k^i the singleton graph whose only vertex is labelled x_j. Then the graphs $G_i := p_i(H_\circ^i, \ldots, H_l^i)$ have the desired property. $\qquad\square$

5.2 Traces and regularisations

Our next goal is to prove that finitary, branch-continuous tree algebras are determined by their regular restrictions. For the proof, we will use the representation of a branch-continuous tree algebra as a quotient of an algebra of the form $\mathrm{Branch}(\mathfrak{S}) = \mathbb{DU}(\mathrm{TA}(\mathfrak{S}))$. We start by recovering the ω-semigroup $\mathfrak{S}$ from $\mathfrak{A}^{\mathrm{reg}}$. In semigroup theory there is a standard way to expand a so-called *Wilke algebra,* the analogue of an RT-algebra, to an ω-semigroup.

Definition 5.5. (a) The functor $\mathbb{W}^{\mathrm{reg}} : \mathsf{pPos} \to \mathsf{pPos}$ is defined by

$$\mathbb{W}^{\mathrm{reg}}_0 A := A_1^{<\omega} A_0 \cup \{\, w \in A_1^\omega \mid w \text{ ultimately periodic} \,\},$$
$$\mathbb{W}^{\mathrm{reg}}_1 A := A_1^{<\omega},$$
$$\mathbb{W}^{\mathrm{reg}}_n A := \varnothing, \quad \text{for } n > 1.$$

(b) An ordered *Wilke algebra* $\langle A, \pi, \leq \rangle$ is a $\mathbb{W}^{\mathrm{reg}}$-algebra $\pi : \mathbb{W}^{\mathrm{reg}} A \to A$.

(c) Given an ω-semigroup $\mathfrak{S}$, we denote by $\mathfrak{S}^{\mathrm{reg}}$ the corresponding Wilke algebra. ⌟

The following is a standard result in the theory of ω-semigroups (see, e.g., Theorem II.5.1 of [8]).

Theorem 5.6. (a) *For every finite Wilke algebra $\mathfrak{S}_0$, there exists a unique ω-semigroup $\mathfrak{S}$ such that $\mathfrak{S}^{\mathrm{reg}} = \mathfrak{S}_0$.*

(b) *Every morphism $\varphi : \mathfrak{S}_0 \to \mathfrak{T}_0$ between finite Wilke algebras is also a morphism $\varphi : \mathfrak{S} \to \mathfrak{T}$ between the corresponding ω-semigroups.*

We will use this theorem to recover the trace ω-semigroup from a RT-algebra.

Definition 5.7. Let $\mathfrak{A}$ be an RT-algebra.

(a) The *Wilke algebra* $\mathrm{SG}^{\mathrm{reg}}(\mathfrak{A})$ *associated* with $\mathfrak{A}$ is the Wilke algebra with domains A_0 and A_1 whose product is inherited from that of $\mathfrak{A}$.

(b) If $\mathfrak{A}$ is finitary, we define the *ω-semigroup* $\mathrm{SG}(\mathfrak{A})$ *associated* with $\mathfrak{A}$ as the unique ω-semigroup whose associated Wilke algebra is equal to $\mathrm{SG}^{\mathrm{reg}}(\mathfrak{A})$. ⌟

Proposition 5.8. *Let $\mathfrak{A}$ be a finitary tree algebra and $S \subseteq A_0 \cup A_1$.*

(a) $\mathrm{SG}(\mathfrak{A}) = \mathrm{SG}(\mathfrak{A}^{\mathrm{reg}})$.

(b) *When computing $\langle S \rangle$ in $\mathfrak{A}$ and $\mathfrak{A}^{\mathrm{reg}}$, we obtain the same result.*

(c) $\mathfrak{B}$ is a semigroup-like subalgebra of $\mathfrak{A}$ if, and only if, $\mathfrak{B}^{\mathrm{reg}}$ is a semigroup-like subalgebra of $\mathfrak{A}^{\mathrm{reg}}$.

(d) For every finitary, semigroup-like RT-algebra $\mathfrak{B}_{\mathrm{o}}$, there exists a unique semigroup-like tree algebra $\mathfrak{B}$ with $\mathfrak{B}^{\mathrm{reg}} = \mathfrak{B}_{\mathrm{o}}$.

Proof. (a) follows from the fact that both ω-semigroups have the same associated Wilke algebra.

(d) follows by Theorem 5.6 and the fact that every semigroup-like tree algebra $\mathfrak{B}$ is uniquely determined by its associated ω-semigroup $\mathrm{SG}(\mathfrak{B})$.

(c) follows by (b).

(b) Let C be the result when computing $\langle S \rangle$ in $\mathfrak{A}^{\mathrm{reg}}$ and let D be the result when computing it in $\mathfrak{A}$. Then

$$C = \mathrm{rng}\,\pi \restriction \mathbb{T}^{\mathrm{reg}} S \quad \text{and} \quad D = \mathrm{rng}\,\pi \restriction \mathbb{T} S .$$

Since $\mathbb{T}^{\mathrm{reg}} S \subseteq \mathbb{T} S$, it follows that $C \subseteq D$.

For the converse, note that π induces an associative function

$$\tilde{\pi} : \mathbb{W}(D_{\mathrm{o}} \cup D_{1}) \to D_{\mathrm{o}} \cup D_{1} ,$$

i.e., an ω-semigroup $\mathfrak{D} = \langle \tilde{D}, \tilde{\pi} \rangle$ with $\tilde{D} := D_{\mathrm{o}} \cup D_{1}$. In the same way, we obtain a Wilke algebra $\mathfrak{C}^{\mathrm{reg}} = \langle C', \tilde{\pi}_{\mathrm{o}} \rangle$ where $C' := C_{\mathrm{o}} \cup C_{1}$ and $\tilde{\pi}_{\mathrm{o}} : \mathbb{W}^{\mathrm{reg}} C' \to C'$. Let $\mathfrak{C} = \langle C', \tilde{\pi}_{1} \rangle$ be the ω-semigroup associated with $\mathfrak{C}^{\mathrm{reg}}$. As every element of C' can be written as a regular product of elements of S, it follows that $\mathfrak{C}^{\mathrm{reg}}$ and, thus, $\mathfrak{C}$ are generated by S. In the same way, we see that $\mathfrak{D}$ is generated by S. Consequently, $D' = C'$, which implies that $D = C$. □

Definition 5.9. Let $\mathfrak{A}$ be a finitary RT-algebra, $\mathfrak{S} \subseteq \mathfrak{A}$ a semigroup-like subalgebra, and $t \in \mathbb{T} A$ a tree.

(a) The *trace set* of t is

$$\mathrm{Tr}_S(t) := \Uparrow \{ \pi(u) \mid u \text{ an } S\text{-trace of } t \} .$$

(b) An *S-regularisation* of t is a regular tree $t_{\mathrm{o}} \in \mathbb{T}^{\mathrm{reg}} A$ such that

$$\mathrm{Tr}_S(t_{\mathrm{o}}) = \mathrm{Tr}_S(t) ,$$

and every label used by t_{o} also occurs somewhere in t. ⌟

Remark. If the tree algebra $\mathfrak{A}$ is branch-continuous with skeleton $\mathfrak{S}$, it follows by Lemma 4.23 that

$$\pi(t) = \inf \mathrm{Tr}_S(t), \quad \text{for all } t \in \mathbb{T}(\mathrm{cl}(S)).$$

As a first application of trace sets, we prove that every finitary RT-algebra can be expanded to a branch-continuous tree algebra in at most one way.

Lemma 5.10. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finitary, branch-continuous tree algebras with skeletons $\mathfrak{S} \subseteq \mathfrak{A}$ and $\mathfrak{T} \subseteq \mathfrak{B}$. Then*

$$\mathfrak{A}^{\mathrm{reg}} = \mathfrak{B}^{\mathrm{reg}} \quad \text{and} \quad S = T \quad \text{implies} \quad \mathfrak{A} = \mathfrak{B}.$$

Proof. Let $\pi : \mathbb{T}A \to A$ be the product of $\mathfrak{A}$ and $\pi' : \mathbb{T}A \to A$ the product of $\mathfrak{B}$. By Proposition 5.8 (d), π and π' agree on trees in $\mathbb{T}S$. As the orderings of $\mathfrak{A}$ and of $\mathfrak{B}$ also coincide, it follows that the closure $C := \mathrm{cl}(S)$ is the same in both algebras. Finally, note that the definition of $\mathrm{Tr}_S(t)$ only depends on $\mathfrak{A}^{\mathrm{reg}} = \mathfrak{B}^{\mathrm{reg}}$. For a tree $t \in \mathbb{T}C$, it therefore follows by Lemma 4.23 that

$$\pi(t) = \inf \mathrm{Tr}_S(t) = \inf \mathrm{Tr}_T(t) = \pi'(t).$$

For an arbitrary term $t \in \mathbb{T}A$, we then have

$$
\begin{aligned}
\pi(t) &= \sup \{ \pi(s) \mid s \in \mathbb{T}C, \ s \leq^{\mathbb{T}} t \} \\
&= \sup \{ \pi'(s) \mid s \in \mathbb{T}C, \ s \leq^{\mathbb{T}} t \} = \pi'(t). \qquad \square
\end{aligned}
$$

To prove the existence of regularisations, we employ a result from [3] on additive labellings.

Definition 5.11. (a) Let $\mathfrak{G} = \langle V, E, v_o \rangle$ be a graph with a distinguished root vertex v_o, let $L \subseteq V$ be the set of leaves of $\mathfrak{G}$, and let $\mathfrak{S}$ be an ω-semigroup. An *additive labelling* of $\mathfrak{G}$ is a function $\lambda : E \cup L \to S$ mapping edges of $\mathfrak{G}$ to unary elements and leaves to 0-ary elements.

(b) For an additive labelling λ of $\mathfrak{G}$ and a (finite or infinite) path $\beta = (e_n)_n$ of $\mathfrak{G}$, we define

$$\lambda(\beta) := \prod_n \lambda(e_n).$$

(If $\beta = e_o \dots e_m$ is finite in the above definition, we allow the last element e_m to be a leaf instead of an edge.) If $\mathfrak{G}$ is a tree and $x \prec y$ are vertices of $\mathfrak{G}$, we also write

$$\lambda(x, y) := \lambda(\beta), \quad \text{where } \beta \text{ is the unique path from } x \text{ to } y.$$

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(b) The *limit set* of λ is

$$\lim \lambda := \{ \lambda(\beta) \mid \beta \text{ a maximal path of } \mathfrak{G} \text{ starting at the root} \} . \qquad \lrcorner$$

The following has been proven in [3].

Theorem 5.12. *Let λ be an additive labelling of a tree t. There exists a finite graph G and an additive labelling λ' of G such that*

$$\lim \lambda = \lim \lambda' \quad and \quad \operatorname{rng} \lambda' \subseteq \operatorname{rng} \lambda .$$

We also need a version for regular trees.

Theorem 5.13. *Let λ be an additive labelling of a regular tree t. There exists a finite graph G and an additive labelling λ' of G such that*

$$\lim \lambda = \lim \lambda' , \quad \operatorname{rng} \lambda' \subseteq \operatorname{rng} \lambda , \quad and \quad \operatorname{un}(G) \simeq_{\mathrm{sh}} t .$$

Proof. (This proof uses terminology and notation from [3].) Let H be a finite graph such that $t = \operatorname{un}(H)$ and let $p : \operatorname{dom}(t) \to \operatorname{dom}(H)$ be the corresponding graph homomorphism. We fix a bijection $\eta : \operatorname{dom}(H) \to [n]$, for some $n < \omega$. Given a Ramseyan split σ of λ, we define a function σ' by

$$\sigma'(v) := n \cdot \sigma(v) + \eta(p(v)) .$$

Since $u \sqsubset_{\sigma'} v$ implies $u \sqsubset_\sigma v$, it follows that σ' is also a Ramseyan split of λ. Let $P \subseteq \operatorname{dom}(t)$ be a set such that

$$\lim \lambda^P_{\sigma'} = \lim \lambda .$$

We claim that the graph $G := \mathfrak{C}^P_{\sigma'}(\lambda)$ and the labelling $\lambda' := \lambda^P_{\sigma'}$ have the desired properties. The inclusion $\operatorname{rng} \lambda' \subseteq \operatorname{rng} \lambda$ holds by definition of λ', and the equation $\lim \lambda' = \lim \lambda$ by choice of P. For the second statement note that, by definition of σ', there exists a graph homomorphism $\varphi : \operatorname{dom}(G) \to \operatorname{dom}(H)$ (ignoring the labelling) which extends to the corresponding unravellings. Consequently, $\operatorname{un}(G) \simeq_{\mathrm{sh}} \operatorname{un}(H) = t$. $\qquad \square$

Remark. In both of the above theorems we can also bound the length of the longest path contained in the graph G. This bound only depends on the size of the ω-semigroup used by λ and, in the second statement, also on the size of the graph H. It does not depend on t.

We use these two theorems to prove the existence of regularisations. To do so, we have to construct suitable additive labellings.

Lemma 5.14. *Let $\mathfrak{A}$ be a finitary RT-algebra, $\mathfrak{S} \subseteq \mathfrak{A}$ a semigroup-like subalgebra, and $n < \omega$. There exists a finite ω-semigroup $\mathfrak{T}$ and a partial function $f : T \to \wp(A_n)$ such that every tree $t \in \mathbb{T}_n(\mathrm{cl}(S))$ has an additive labelling λ_t over $\mathfrak{T}$ with*

$$\mathrm{Tr}_S(t) = \bigcup f[\lim \lambda_t] \, .$$

Proof. Set $\mathfrak{T}' := \mathrm{SG}(\mathfrak{S})$. The desired ω-semigroup $\mathfrak{T}$ is derived from the tree algebra $\mathbb{UTA}(\mathfrak{T}')$ as follows. The domains are

$$T_0 := \mathbb{U}T_0' \cup \left((\mathbb{U}T_1' \cup \{1\}) \times [n] \right) \quad \text{and} \quad T_1 := \mathbb{U}T_1' \, .$$

The product of $\mathfrak{T}$ extends that of $\mathbb{U}\mathfrak{T}'$ by

$$I \cdot \langle J, k \rangle := \langle IJ, k \rangle \, , \quad \text{for } I \in \mathbb{U}T_1' \text{ and } \langle J, k \rangle \in (\mathbb{U}T_1' \cup \{1\}) \times [n]$$

(where $I \cdot 1 := I$). We use the partial function $f : T \to \wp(A_n)$ defined by

$$
\begin{aligned}
f(I) &:= I \, , & &\text{for } I \in \mathbb{U}T_0' \cup \mathbb{U}T_1' \, , \\
f(\langle J, k \rangle) &:= J(x_k) \, , & &\text{for } \langle J, k \rangle \in \mathbb{U}T_1' \times [n] \, , \\
f(\langle 1, k \rangle) & \text{ is undefined.}
\end{aligned}
$$

Finally, given a tree $t \in \mathbb{T}_n(\mathrm{cl}(S))$, we define the desired additive labelling λ_t over $\mathfrak{T}$ by

$$
\begin{aligned}
\lambda_t(v, vk) &:= \{\, c \in S_0 \cup S_1 \mid \mathrm{cy}_k(c) \geq t(v) \,\} \, , \\
& \hspace{3.5cm} \text{for } v \in \mathrm{dom}(t) \text{ and } k < \mathrm{ar}(t(v)), \\
\lambda_t(v) &:= \{\, c \in S_0 \mid c \geq t(v) \,\} \, , & &\text{for leaves } v \in \mathrm{dom}(t) \smallsetminus \mathrm{Hole}(t), \\
\lambda_t(v) &:= \langle 1, k \rangle \, , & &\text{for holes } v = \mathrm{hole}_k(t).
\end{aligned}
$$

Then it follows for an S-trace u of t along some branch β that

$$u(0^n) \in \lambda_t(\beta \restriction n, \, \beta \restriction (n+1)) \, .$$

Hence,

$$\pi(u) \in \bigcup f(\lambda_t(\beta)) \, .$$

Consequently, $\mathrm{Tr}_S(t) = \bigcup f[\lim \lambda_t]$. $\qquad\square$

Theorem 5.15. *Let $\mathfrak{A}$ be a finitary RT-algebra and $\mathfrak{S} \subseteq \mathfrak{A}$ a semigroup-like subalgebra.*

(a) *Every tree $t \in \mathbb{T}A$ has an S-regularisation $t_o \in \mathbb{T}^{\mathrm{reg}}A$.*

(b) *If there exists some function $f : A \to B$ such that $\mathbb{T}f(t)$ is regular, then we can choose the S-regularisation t_o such that $t_o \simeq_{\mathrm{sh}} t$.*

Proof. Given a tree $t \in \mathbb{T}_n A$, we use the labelling λ_t over the ω-semigroup $\mathfrak{T}$ from Lemma 5.14 to find an S-regularisation of t as follows. By Theorem 5.12, there exists a finite graph G and an additive labelling λ_G of G such that $\lim \lambda_G = \lim \lambda_t$ and $\mathrm{rng}\, \lambda_G \subseteq \mathrm{rng}\, \lambda_t$. For (b), we can use Theorem 5.13 to ensure that $\mathrm{un}(G) \simeq_{\mathrm{sh}} t$. Let $t_o \in \mathbb{T}_n^{\mathrm{reg}}A$ be a regular tree such that $t_o \simeq_{\mathrm{sh}} \mathrm{un}(G)$ and the labelling λ_{t_o} associated with t_o coincides with the (unravelling of) λ_G. Then

$$\mathrm{Tr}_S(t) = \bigcup f[\lim \lambda_t] = \bigcup f[\lim \lambda_{t_o}] = \mathrm{Tr}_S(t_o).$$

Moreover, in case (b) we have $t_o \simeq_{\mathrm{sh}} \mathrm{un}(G) \simeq_{\mathrm{sh}} t$.

Hence, it remains to prove that every label used in t_o also occurs in t. The construction above does not yield this fact. We have to modify it slightly by changing the labelling λ such that the value $\lambda(x, y)$ also encodes the label $t(x)$ (say, by using a suitable ω-semigroup with domain $A \times T$ and setting $\lambda_t'(x, y) := \langle t(x), \lambda_t(x, y) \rangle$). Then the claim follows from the condition that $\mathrm{rng}\, \lambda_G \subseteq \mathrm{rng}\, \lambda_t$. $\square$

Existence of regularisations can be strengthened in the following way.

Lemma 5.16. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finitary, branch-continuous tree algebras, $\mathfrak{S} \subseteq \mathfrak{A}$ and $\mathfrak{T} \subseteq \mathfrak{B}$ corresponding skeletons, and $\theta \subseteq A \times B$ a binary relation. For every pair of trees $s \in \mathbb{T}A$ and $t \in \mathbb{T}B$ with $s \, \theta^{\mathbb{T}} \, t$, there are an S-regularisation s_o of s and a T-regularisation t_o of t such that $s_o \, \theta^{\mathbb{T}} \, t_o$.*

Proof. $s \, \theta^{\mathbb{T}} \, t$ implies $s \simeq_{\mathrm{sh}} t$. Hence, there exists a tree $u \in \mathbb{T}(A \times B)$ such that $s = \mathbb{T}p(u)$ and $t = \mathbb{T}q(u)$, where $p : A \times B \to A$ and $q : A \times B \to B$ are the two projection functions. By Theorem 4.33, the product $\mathfrak{A} \times \mathfrak{B}$ is finitary and branch-continuous with skeleton $\mathfrak{S} \times \mathfrak{T}$. Consequently, we can use Theorem 5.15 to find an $(S \times T)$-regularisation u_o of u. Set $s_o := \mathbb{T}p(u_o)$ and $t_o := \mathbb{T}q(u_o)$. Then

$$\mathrm{Tr}_S(s_o) = p[\mathrm{Tr}_S(u_o)] = p[\mathrm{Tr}_S(u)] = \mathrm{Tr}_S(s)$$
$$\text{and} \quad \mathrm{Tr}_S(t_o) = q[\mathrm{Tr}_S(u_o)] = q[\mathrm{Tr}_S(u)] = \mathrm{Tr}_S(t).$$

Hence, s_o is an S-regularisation of s and t_o is a T-regularisation of t. Furthermore,

$$s(v)\ \theta\ t(v) \quad\text{implies}\quad u(v)\in\theta\,, \quad\text{for all } v\in\mathrm{dom}(u)\,.$$

As all labels used by u_o also appear in u, we have

$$u_\mathrm{o}(v)\in\theta\,, \quad\text{which implies that}\quad s_\mathrm{o}(v)\ \theta\ t_\mathrm{o}(v)\,, \quad\text{for all } v\in\mathrm{dom}(u)\,.$$

Consequently, $s_\mathrm{o}\ \theta^{\mathbb{T}}\ t_\mathrm{o}$. $\qquad\qquad\square$

5.3 Expansion of the regular product

We have already shown in Lemma 5.10 that an RT-algebra can be expanded to at most one full tree algebra. In general such an expansion does not need to exist, but it does in the case of algebras that are finitary and branch-continuous. Let us start by defining branch-continuity for RT-algebras.

Definition 5.17. A RT-algebra $\mathfrak{A}$ is *branch-continuous* if it is complete, distributive, and it has a semigroup-like subalgebra $\mathfrak{S}\subseteq\mathfrak{A}$ with the following properties.

◆ $C \coloneqq \mathrm{cl}(S)$ is a set of join-generators of $\mathfrak{A}$.

◆ $\mathrm{SG}(\mathfrak{S})$ is meet-continuous.

◆ For every tree $U\in\mathbb{TP}(S)$,

$$\sup\big\{\inf\mathrm{Tr}_S(s) \;\big|\; s\in\mathbb{T}C,\, s\le^{\mathbb{T}}\mathbb{T}\inf(U)\big\} = \inf\big\{\inf\mathrm{Tr}_S(s) \;\big|\; s\in^{\mathbb{T}}U\big\}\,.$$

◆ For every regular tree $t\in\mathbb{T}^{\mathrm{reg}}A$,

$$\pi(t) = \sup\big\{\inf\mathrm{Tr}_S(s) \;\big|\; s\in\mathbb{T}C \text{ with } s\le^{\mathbb{T}}t\big\}\,.$$

◆ For arbitrary trees $U,U'\in\mathbb{TD}C$ with $\mathbb{T}\sup(U)=\mathbb{T}\sup(U')$,

$$\sup\big\{\inf\mathrm{Tr}_S(s) \;\big|\; s\in^{\mathbb{T}}U\big\} = \sup\big\{\inf\mathrm{Tr}_S(s') \;\big|\; s'\in^{\mathbb{T}}U'\big\}\,.$$

◆ For arbitrary trees $U,U'\in\mathbb{TU}S$ with $\mathbb{T}\inf(U)=\mathbb{T}\inf(U')$,

$$\inf\big\{\inf\mathrm{Tr}_S(s) \;\big|\; s\in^{\mathbb{T}}U\big\} = \inf\big\{\inf\mathrm{Tr}_S(s') \;\big|\; s'\in^{\mathbb{T}}U'\big\}\,.$$

Such a subalgebra $\mathfrak{S}$ is called a *skeleton* of $\mathfrak{A}$.

Lemma 5.18. *Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a surjective morphism of RT-algebras that preserves arbitrary meets and joins.*

(a) *If $\mathfrak{S}$ is a skeleton of $\mathfrak{A}$, then $\varphi[\mathfrak{S}]$ is a one of $\mathfrak{B}$.*

(b) *If $\mathfrak{A}$ is branch-continuous, then so is $\mathfrak{B}$.*

Proof. (a) Let $T := \varphi[S]$ and $D := \varphi[C]$ where $C := \mathrm{cl}(S)$. All conditions in the definition of a skeleton take the form of an equation between terms involving meets, joins, and products. Every equation of this form is preserved by φ.

(b) According to Lemma 2.14, $\mathfrak{B}$ is complete and distributive. Hence, the claim follows by (a). □

Lemma 5.19. *Let $\mathfrak{A}$ be a finitary, branch-continuous tree algebra and $\mathfrak{S}$ a skeleton of $\mathfrak{A}$. Then $\mathfrak{A}^{\mathrm{reg}}$ is branch-continuous and $\mathfrak{S}^{\mathrm{reg}}$ is a skeleton of $\mathfrak{A}^{\mathrm{reg}}$.*

Proof. First note that $\mathfrak{A}^{\mathrm{reg}}$ is complete and distributive since these two properties are defined solely in terms of the ordering. Hence, it remains to prove that $\mathfrak{S}^{\mathrm{reg}}$ is a skeleton of $\mathfrak{A}^{\mathrm{reg}}$. Clearly, the set $C := \mathrm{cl}(S)$ is a set of join-generators. Hence, it remains to prove the following ones.

(a) $\mathrm{TS}(S)$ is meet-continuous.

(b) For every tree $U \in \mathbb{T}\wp(S)$,

$$\sup \left\{ \inf \mathrm{Tr}_S(s) \mid s \in \mathbb{T}C, \, s \leq^{\mathbb{T}} \mathbb{T}\inf(U) \right\} = \inf \left\{ \inf \mathrm{Tr}_S(s) \mid s \in^{\mathbb{T}} U \right\}.$$

(c) For every regular tree $t \in \mathbb{T}^{\mathrm{reg}}A$,

$$\pi(t) = \sup \left\{ \inf \mathrm{Tr}_S(s) \mid s \in \mathbb{T}C \text{ with } s \leq^{\mathbb{T}} t \right\}.$$

(d) For arbitrary trees $U, U' \in \mathbb{T}\mathbb{D}C$ with $\mathbb{T}\sup(U) = \mathbb{T}\sup(U')$,

$$\sup \left\{ \inf \mathrm{Tr}_S(s) \mid s \in^{\mathbb{T}} U \right\} = \sup \left\{ \inf \mathrm{Tr}_S(s') \mid s' \in^{\mathbb{T}} U' \right\}.$$

(e) For arbitrary trees $U, U' \in \mathbb{T}\mathbb{U}S$ with $\mathbb{T}\inf(U) = \mathbb{T}\inf(U')$,

$$\inf \left\{ \inf \mathrm{Tr}_S(s) \mid s \in^{\mathbb{T}} U \right\} = \inf \left\{ \inf \mathrm{Tr}_S(s') \mid s' \in^{\mathbb{T}} U' \right\}.$$

(a) follows from Proposition 5.8 and the fact that $\mathfrak{S}$ is meet-continuously embedded in $\mathfrak{A}$.

For (b), let $U \in \mathbb{TP}(S)$. Since $\mathfrak{S}$ is meet-continuously embedded in $\mathfrak{A}$, we have

$$\sup \big\{ \inf \mathrm{Tr}_S(s) \,\big|\, s \in \mathbb{T}C, \ s \leq^{\mathbb{T}} \mathbb{T}\inf(U) \big\}$$
$$= \pi(\mathbb{T}\inf(U)) = \inf \{ \pi(s) \mid s \in^{\mathbb{T}} U \} = \inf \{ \inf \mathrm{Tr}_S(s) \mid s \in^{\mathbb{T}} U \}.$$

(c) follows from join-continuity of $\mathfrak{A}$ and the fact that $\pi(s) = \inf \mathrm{Tr}_S(s)$, for trees $t \in \mathbb{T}C$.

For (d), consider two trees U and U' as above. Setting $t := \mathbb{T}\sup(U)$, it follows by join-continuity that

$$\sup \big\{ \inf \mathrm{Tr}_S(s) \,\big|\, s \in^{\mathbb{T}} U \big\} = \pi(t) = \sup \big\{ \inf \mathrm{Tr}_S(s') \,\big|\, s' \in^{\mathbb{T}} U' \big\}.$$

(e) follows in the same way as (d) using the fact that $\mathfrak{S}$ is meet-continuously embedded in $\mathfrak{A}$. $\qquad\square$

We will expand a finitary, branch-continuous RT-algebra to a full tree algebra in two steps. We first define the full product on the set $\mathrm{cl}(S)$; then we extend it to the whole algebra.

Lemma 5.20. *Let* $\mathfrak{A}_o = \langle A, \pi_o, \leq \rangle$ *be a finitary, branch-continuous RT-algebra,* $\mathfrak{S}_o = \langle S, \pi_o, \leq \rangle$ *a skeleton of* $\mathfrak{A}_o$, *and let* $C := \mathrm{cl}(S)$. *Define* $\pi : \mathbb{T}C \to C$ *by*

$$\pi(t) := \inf \mathrm{Tr}_S(t), \quad \text{for } t \in \mathbb{T}C.$$

Then $\mathfrak{C} := \langle C, \pi, \leq \rangle$ *is a tree algebra such that* $\mathfrak{C}^{\mathrm{reg}} \subseteq \mathfrak{A}_o$.

Proof. The function π extends π_o since, for a regular term $t \in \mathbb{T}^{\mathrm{reg}}C$, we have

$$\pi_o(t) = \sup \big\{ \inf \mathrm{Tr}_S(s) \,\big|\, s \in \mathbb{T}C, \ s \leq^{\mathbb{T}} t \big\} = \inf \mathrm{Tr}_S(t) = \pi(t).$$

Hence, it remains to prove that it forms a tree algebra. First, note that, according to Proposition 5.8, there exists a unique semigroup-like tree algebra $\mathfrak{S} = \langle S, \pi_1, \leq \rangle$ with $\mathfrak{S}^{\mathrm{reg}} = \mathfrak{S}_o$. Applying Proposition 4.11 to the tree algebra $\mathfrak{S}$, it further follows that its product $\pi_1 : \mathbb{T}S \to S$ takes the form

$$\pi(t) = \inf \mathrm{Tr}_S(t), \quad \text{for } t \in \mathbb{T}S.$$

One of the axioms of a skeleton states that this function π_1 satisfies the meet-extension condition. Therefore, we can apply Proposition 3.24 to the embedding

$S \to A$, and it follows that there exists a tree algebra $\mathfrak{C} = \langle C, \pi', \le \rangle$ where the product $\pi' : \mathbb{T}C \to C$ extends π_o and it is given by

$$\pi'(t) := \inf \left\{ \pi_1(s) \mid s \in \mathbb{T}S, \, s \ge^\mathbb{T} t \right\}.$$

We claim that $\pi' = \pi$. For $t \in \mathbb{T}C$, we have

$$
\begin{aligned}
\pi'(t) &= \inf \left\{ \pi_1(s) \mid s \in \mathbb{T}S, \, s \ge^\mathbb{T} t \right\} \\
&= \inf \left\{ \inf \mathrm{Tr}_S(s) \mid s \in \mathbb{T}S, \, s \ge^\mathbb{T} t \right\} \\
&= \inf \bigcup \left\{ \mathrm{Tr}_S(s) \mid s \in \mathbb{T}S, \, s \ge^\mathbb{T} t \right\} \\
&= \inf \left\{ \pi(u) \mid s \in \mathbb{T}S, \, s \ge^\mathbb{T} t, \, u \text{ an } S\text{-trace of } s \right\} \\
&= \inf \left\{ \pi(u) \mid u \text{ an } S\text{-trace of } t \right\} \\
&= \inf \mathrm{Tr}_S(t) \\
&= \pi(t).
\end{aligned}
$$

Consequently, $\pi = \pi' : \mathbb{T}C \to C$ is a tree algebra. $\qquad\square$

Proposition 5.21. *Let $\mathfrak{A}_\mathrm{o} = \langle A, \pi_\mathrm{o}, \le \rangle$ be a finitary, branch-continuous RT-algebra and $\mathfrak{S}_\mathrm{o} = \langle S, \pi_\mathrm{o}, \le \rangle$ a skeleton of $\mathfrak{A}_\mathrm{o}$. There exists a finitary, branch-continuous tree algebra $\mathfrak{A}$ with $\mathfrak{A}^\mathrm{reg} = \mathfrak{A}_\mathrm{o}$.*

Proof. Let $C := \mathrm{cl}(S)$. In Lemma 5.20 we seen that the function $\pi_1 : \mathbb{T}C \to C$ with

$$\pi_1(t) := \inf \mathrm{Tr}_S(t), \quad \text{for } t \in \mathbb{T}C,$$

is the product of a tree algebra extending π_o on C. One of the axioms of a skeleton states that π_1 satisfies the join-extension condition. Therefore, we can apply Proposition 3.17 to the embedding $C \to A$, and it follows that there exists a tree algebra $\mathfrak{A} = \langle A, \pi, \le \rangle$ where the product $\pi : \mathbb{T}A \to A$ is given by

$$\pi(t) := \sup \left\{ \pi_1(s) \mid s \in \mathbb{T}C, \, s \le^\mathbb{T} t \right\}.$$

To prove that π extends π_o, consider a regular tree $t \in \mathbb{T}^\mathrm{reg} A$. Since S is a skeleton of $\mathfrak{A}_\mathrm{o}$, we have

$$
\begin{aligned}
\pi(t) &= \sup \left\{ \pi_1(s) \mid s \in \mathbb{T}C, \, s \le^\mathbb{T} t \right\} \\
&= \sup \left\{ \inf \mathrm{Tr}_S(s) \mid s \in \mathbb{T}C, \, s \le^\mathbb{T} t \right\} = \pi_\mathrm{o}(t).
\end{aligned}
$$

We have shown that $\mathfrak{A}$ is a tree algebra extending $\mathfrak{A}_0$. Furthermore, it is clearly finitary, complete, and distributive as these properties transfer from $\mathfrak{A}_0$. For join-continuity, suppose that $t = \mathbb{T} \sup U$ for $t \in \mathbb{T}A$ and $U \in \mathbb{T}\wp(A)$. Let $U', R \simeq_{\mathrm{sh}} U$ be the trees with

$$U'(v) := C \cap {\Downarrow} U(v) \quad \text{and} \quad R(v) := C \cap {\Downarrow} t(v), \quad \text{for all } v.$$

Then $\sup U'(v) = \sup U(v) = t(v) = \sup R(v)$. Since S is a skeleton of $\mathfrak{A}_0$, it follows that

$$
\begin{aligned}
\pi(t) &= \sup \left\{ \pi_1(r) \mid r \in^{\mathbb{T}} R \right\} \\
&= \sup \left\{ \pi_1(r) \mid r \in^{\mathbb{T}} U' \right\} \\
&= \sup \left\{ \pi_1(r) \mid r \in \mathbb{T}C,\ r \leq^{\mathbb{T}} s \text{ for some } s \in^{\mathbb{T}} U \right\} \\
&= \sup \left\{ \sup \{ \pi_1(r) \mid r \in \mathbb{T}C,\ r \leq^{\mathbb{T}} s \} \mid s \in^{\mathbb{T}} U \right\} \\
&= \sup \left\{ \pi(s) \mid s \in^{\mathbb{T}} U \right\}.
\end{aligned}
$$

It remains to check branch-continuity. We claim that S is a skeleton of $\mathfrak{A}$. Clearly, $C = \mathrm{cl}(S)$ is a set of join-generators of $\mathfrak{A}$. By definition of π_1, we furthermore have

$$\pi(t) = \pi_1(t) = \inf \mathrm{Tr}_S(t), \quad \text{for } t \in \mathbb{T}C.$$

Hence, we only have to show that S is meet-continuously embedded in $\mathfrak{A}$. Let $t \in \mathbb{T}A$ and $U \in \mathbb{T}\wp(S)$ be trees such that $t = \mathbb{T}\inf(U)$. Then

$$
\begin{aligned}
\pi(t) &= \sup \left\{ \inf \mathrm{Tr}_S(s) \mid s \in \mathbb{T}C,\ s \leq^{\mathbb{T}} t \right\} \\
&= \inf \left\{ \inf \mathrm{Tr}_S(s) \mid s \in^{\mathbb{T}} U \right\} = \inf \left\{ \pi(s) \mid s \in^{\mathbb{T}} U \right\}. \qquad \square
\end{aligned}
$$

Summarising our results, we have obtained the following theorem.

Theorem 5.22. *For every finitary, branch-continuous RT-algebra $\mathfrak{A}_0$, there exists a unique branch-continuous tree algebra $\mathfrak{A}$ with $\mathfrak{A}^{\mathrm{reg}} = \mathfrak{A}_0$.*

Proof. Uniqueness follows by Lemma 5.10 and existence by Proposition 5.21. $\square$

There is a similar statement for morphisms instead of algebras.

Proposition 5.23. *Let $\varphi : \mathfrak{A}^{\mathrm{reg}} \to \mathfrak{B}^{\mathrm{reg}}$ be a surjective morphism between RT-algebras that preserves meets and joins. If $\mathfrak{A}^{\mathrm{reg}}$ is finitary and branch-continuous, then so is $\mathfrak{B}^{\mathrm{reg}}$ and φ is a morphism $\mathfrak{A} \to \mathfrak{B}$ between the corresponding tree algebras.*

Proof. Let $\mathfrak{S}$ be a skeleton of $\mathfrak{A}^{\mathrm{reg}}$ and set $C := \mathrm{cl}(S)$. According to Lemma 5.18, $\mathfrak{B}^{\mathrm{reg}}$ is branch-continuous and the image $\mathfrak{T} := \varphi[\mathfrak{S}]$ is a skeleton of $\mathfrak{B}^{\mathrm{reg}}$. Hence, it remains to show that $\varphi \circ \pi = \pi \circ \mathbb{T}\varphi$. To do so it is sufficient to prove that

$$\varphi(\pi(t)) = \pi(\mathbb{T}\varphi(t)), \quad \text{for all } t \in \mathbb{T}C.$$

Since $\mathbb{T}C$ is a set of join-generators of $\mathbb{T}\mathfrak{A}$, it then follows by Lemma 2.19 (b) that $\varphi \circ \pi = \pi \circ \mathbb{T}\varphi$.

To prove the claim, let $t \in \mathbb{T}C$. By definition of C, there is a tree $T \in \mathbb{T}\mathbb{U}S$ such that $t = \mathbb{T}\inf(T)$. Setting $T' := \mathbb{T}\mathbb{U}\varphi(T)$, it follows for $v \in \mathrm{dom}(T)$ that

$$\inf T'(v) = \inf \Uparrow \varphi[T(v)] = \varphi(\inf T(v)) = \varphi(t(v)).$$

Moreover, note that, since $\mathfrak{S}$ is semigroup-like, Theorem 5.6 implies that

$$\varphi(\pi(t)) = \pi(\mathbb{T}\varphi(t)), \quad \text{for all } t \in \mathbb{T}S.$$

Hence, meet-continuity implies that

$$\begin{aligned}
\pi(\mathbb{T}\varphi(t)) &= \inf \{\, \pi(s') \mid s' \in^{\mathbb{T}} T' \,\} \\
&= \inf \{\, \pi(s') \mid s' \in^{\mathbb{T}} \mathbb{T}\mathbb{U}\varphi(T) \,\} \\
&= \inf \{\, \pi(\mathbb{T}\varphi(s)) \mid s \in^{\mathbb{T}} T \,\} \\
&= \inf \{\, \varphi(\pi(s)) \mid s \in^{\mathbb{T}} T \,\} = \varphi(\pi(t)). \qquad \square
\end{aligned}$$

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