

IA168 Algorithmic Game Theory

Tomáš Brázdil

Organization of This Course

Sources:

- ▶ Lectures (slides, notes)
 - ▶ based on several sources
 - ▶ slides are prepared for lectures, some stuff on greenboard
(\Rightarrow attend the lectures)

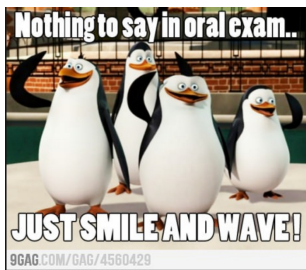
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- ▶ Books:
 - ▶ Nisan/Roughgarden/Tardos/Vazirani, **Algorithmic Game Theory**, Cambridge University, 2007.
Available online for free:
http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf
 - ▶ Tadelis, **Game Theory: An Introduction**, Princeton University Press, 2013

(I use various resources, so please, attend the lectures)

- ▶ Oral exam
- ▶ Homework



- ▶ 3 homework assignments

Notable features of the course

- ▶ No computer games course!
- ▶ **Very demanding!**
- ▶ Mathematical!

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An example of an instruction email (from another course with the same system):

It is typically not sufficient to devote a single afternoon to the preparation for the exam.

You have to know `_everything_` (which means every single thing) starting with the slide 42 and ending with the slide 245 with notable exceptions of slides: 121 - 123, 137 - 140, 165, 167.

Proofs presented on the whiteboard are also mandatory.

Most importantly,

The previous slide is not
a joke!

What is Algorithmic Game Theory?

First, what is the game theory?

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What does the "algorithmic" mean?

- ▶ It means that we are "concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."

Let's have a look at some examples

Prisoner's Dilemma

Prisoners' dilemma



		prisoner B			
		confess		remain silent	
prisoner A	confess	 5 years 5 years	 0 year 20 years		
	remain silent	 20 years 0 year	 1 year 1 year		

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The problem: What would the suspects do?

Prisoner's Dilemma – Solution(?)

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C	-5, -5	0, -20
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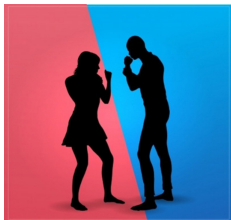
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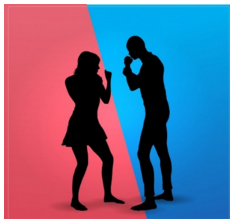
Are there always "dominant" strategies?

Nash equilibria – Battle of Sexes



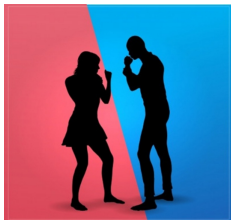
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- ▶ One of them wants to go to the football game. The other one to the opera. Both would prefer to go to the same place rather than different ones.

If they cannot communicate, where should they go?

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Battle of Sexes can be modeled as a game of two players (the couple) with the following payoffs:

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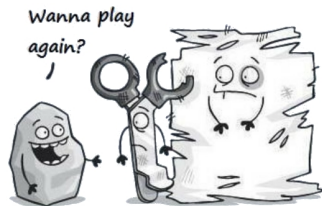
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(O, O) is an example of a *Nash equilibrium* (as is (F, F))

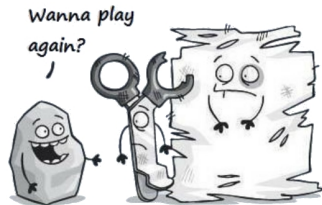
Mixed Equilibria – Rock-Paper-Scissors

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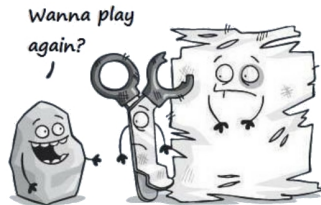
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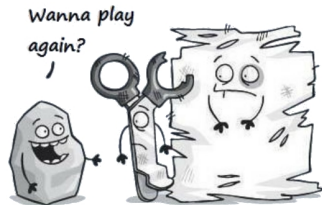
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Use *mixed strategies*: Each player plays each pure strategy with probability $1/3$. The expected payoff of each player is 0 (even if one of the players changes his strategy, he still gets 0!).

Philosophical Issues in Games

I UNDERSTAND THAT SCISSORS CAN BEAT PAPER, AND I GET HOW ROCK CAN BEAT SCISSORS, BUT THERE'S NO WAY PAPER CAN BEAT ROCK. PAPER IS SUPPOSED TO MAGICALLY WRAP AROUND ROCK LEAVING IT IMMOBILE? WHY CAN'T PAPER DO THIS TO SCISSORS? SCREW SCISSORS, WHY CAN'T PAPER DO THIS TO PEOPLE? WHY AREN'T SHEETS OF COLLEGE RULED NOTEBOOK PAPER CONSTANTLY SUFFOCATING STUDENTS AS THEY ATTEMPT TO TAKE NOTES IN CLASS? I'LL TELL YOU WHY, BECAUSE PAPER CAN'T BEAT ANYBODY, A ROCK WOULD TEAR IT UP IN TWO SECONDS. WHEN I PLAY ROCK PAPER SCISSORS, I ALWAYS CHOOSE ROCK. THEN WHEN SOMEBODY CLAIMS TO HAVE BEATEN ME WITH THEIR PAPER I CAN PUNCH THEM IN THE FACE WITH MY ALREADY CLENCHED FIST AND SAY, OH SORRY, I THOUGHT PAPER WOULD PROTECT YOU.

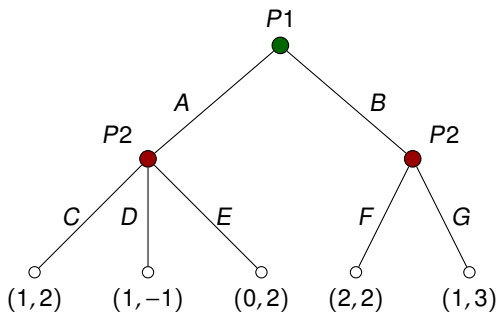
Dynamic Games

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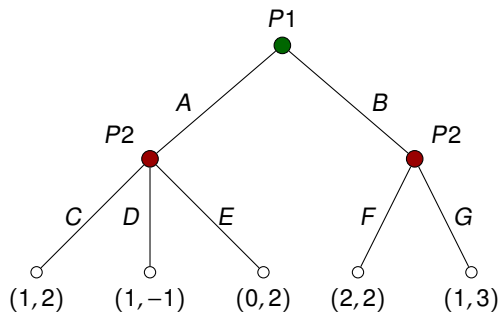
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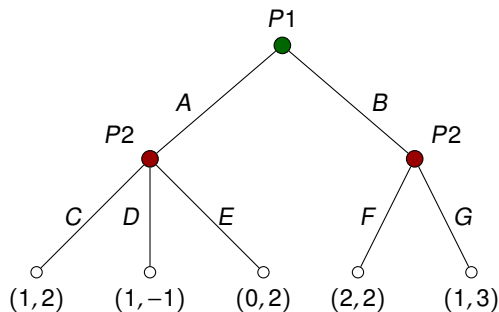


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What is their relationship to the strategic form games?

Chance and Imperfect Information

Some decisions in the game tree may be by chance and controlled by neither player (e.g. Poker, Backgammon, etc.)

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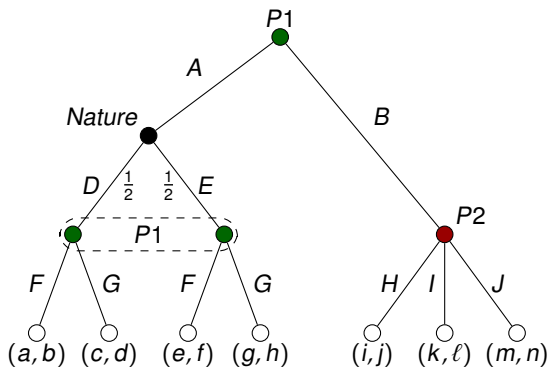
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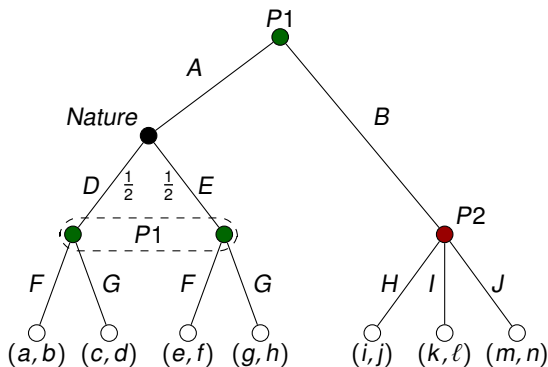
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Again, how to solve such games?

Games of Incomplete Information

In all previous games the players knew all details of the game they played, and this fact was a “common knowledge”. This is not always the case.

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$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

Here v_1 is the private value that player 1 assigns to the item and so the player 2 **does not know** u_1 .

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How to deal with such a game? Assume the “worst” private value?
What if we have a partial knowledge about the private values?

Inefficiency of Equilibria

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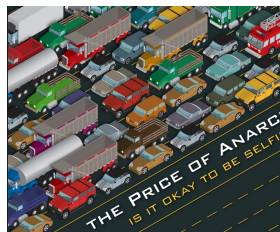
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Price of Anarchy is the maximum ratio between values of equilibria and the value of an optimal solution.

Inefficiency of Equilibria – Selfish Routing

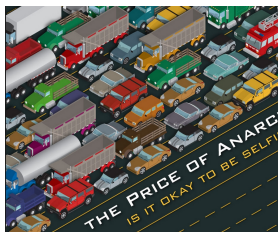
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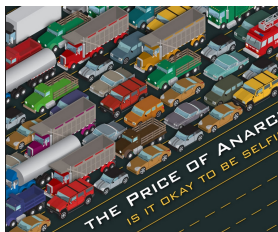
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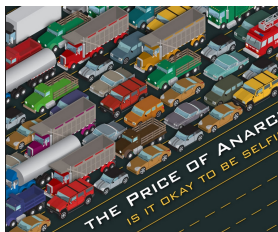


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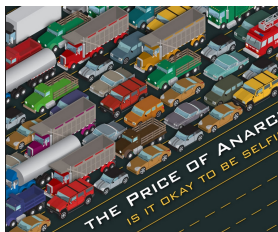
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Price of Anarchy measure the ratio between average travel time in these two cases.

Problem: Bound the price of anarchy over all routing games?



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- ▶ Games in Logic: modal and temporal logics, Ehrenfeucht-Fraisse games, etc.

Games, the Internet and E-commerce: An extremely active research area at the intersection of CS and Economics

Basic idea: “The internet is a HUGE experiment in interaction between agents (both human and automated)”

How do we set up the rules of this game to harness “socially optimal” results?

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- ▶ Remaining time will be devoted to selected topics from extensive form games, games on graphs etc.

Static Games of Complete Information

Strategic-Form Games

Solution concepts

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Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.

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A fact E is a *common knowledge* among players $\{1, \dots, n\}$ if for every sequence $i_1, \dots, i_k \in \{1, \dots, n\}$ we have that i_1 knows that i_2 knows that ... i_{k-1} knows that i_k knows E .

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The goal of each player is to maximize his payoff (and this fact is a common knowledge).

Strategic-Form Games

To formally represent static games of complete information we define *strategic-form games*.

Definition 2

A game in *strategic-form* (or normal-form) is an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, in which:

- ▶ $N = \{1, 2, \dots, n\}$ is a finite set of *players*.
- ▶ S_i is a set of (*pure*) *strategies* of player i , for every $i \in N$.

A *strategy profile* is a vector of strategies of all players $(s_1, \dots, s_n) \in S_1 \times \dots \times S_n$.

We denote the set of all strategy profiles by $S = S_1 \times \dots \times S_n$.

- ▶ $u_i : S \rightarrow \mathbb{R}$ is a function associating each strategy profile $s = (s_1, \dots, s_n) \in S$ with the *payoff* $u_i(s)$ to player i , for every player $i \in N$.

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Definition 3

A *zero-sum* game G is one in which for all $s = (s_1, \dots, s_n) \in S$ we have $u_1(s) + u_2(s) + \dots + u_n(s) = 0$.

Example: Prisoner's Dilemma

- ▶ $N = \{1, 2\}$
- ▶ $S_1 = S_2 = \{S, C\}$
- ▶ u_1, u_2 are defined as follows:
 - ▶ $u_1(C, C) = -5, u_1(C, S) = 0, u_1(S, C) = -20,$
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We usually write payoffs in the following form:

	C	S
C	-5, -5	0, -20
S	-20, 0	-1, -1

or as two matrices:

	C	S
C	-5	0
S	-20	-1

	C	S
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Strategic-form game model $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- ▶ $N = \{1, 2\}$
- ▶ $S_i = [0, \infty)$
- ▶ $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1$
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Solution Concepts

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(I follow the approach of Steven Tadelis here, it is not completely standard)

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Example 4

Nash equilibrium is a solution concept. That is, we “solve” games by finding Nash equilibria and declare them to be reasonable outcomes.

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Throughout the lecture we assume that:

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Here 4. implies non-cooperative game theory: Each player is in control of his actions, and he will stick to an action only if he finds it to be in his best interest.

Evaluating Solution Concepts

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For now, let us concentrate on

pure strategies only!

I.e., no mixed strategies are allowed. We will generalize to mixed setting later.

- ▶ Let $N = \{1, \dots, n\}$ be a finite set and for each $i \in N$ let X_i be a set. Let $X := \prod_{i \in N} X_i = \{(x_1, \dots, x_n) \mid x_j \in X_j, j \in N\}$.
 - ▶ For $i \in N$ we define $X_{-i} := \prod_{j \neq i} X_j$, i.e.,

$$X_{-i} = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mid x_j \in X_j, \forall j \neq i\}$$

- ▶ An element of X_{-i} will be denoted by

$$x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

We slightly abuse notation and write (x_i, x_{-i}) to denote $(x_1, \dots, x_i, \dots, x_n) \in X$.

Strict Dominance in Pure Strategies

Definition 5

Let $s_i, s'_i \in S_i$ be strategies of player i . Then s'_i is *strictly dominated* by s_i (write $s_i \succ s'_i$) if for any possible profile of the other players' strategies, $s_{-i} \in S_{-i}$, we have

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Claim 1

An intelligent and rational player will never play a strictly dominated strategy.

Clearly, intelligence implies that the player should recognize dominated strategies, rationality implies that the player will avoid playing them.

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$s_i \in S_i$ is *strictly dominant* if every other pure strategy of player i is strictly dominated by s_i .

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Corollary 8

If the strictly dominant strategy equilibrium exists, it is unique and rational players will play it.

Examples

In the Prisoner's dilemma:

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<i>C</i>	-5, -5	0, -20
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In the Battle of Sexes:

	<i>O</i>	<i>F</i>
<i>O</i>	2, 1	0, 0
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no strictly dominant strategies exist.

Indiana Jones and the Last Crusade

(Taken from Dixit & Nalebuff's "The Art of Strategy" and a lecture of Robert Marks)

Indiana Jones, his father, and the Nazis have all converged at the site of the Holy Grail. The two Joneses refuse to help the Nazis reach the last step. So the Nazis shoot Indiana's dad. Only the healing power of the Holy Grail can save the senior Dr. Jones from his mortal wound. Suitably motivated, Indiana leads the way to the Holy Grail. But there is one final challenge. He must choose between literally scores of chalices, only one of which is the cup of Christ. While the right cup brings eternal life, the wrong choice is fatal. The Nazi leader impatiently chooses a beautiful gold chalice, drinks the holy water, and dies from the sudden death that follows from the wrong choice. Indiana picks a wooden chalice, the cup of a carpenter. Exclaiming "There's only one way to find out" he dips the chalice into the font and drinks what he hopes is the cup of life. Upon discovering that he has chosen wisely, Indiana brings the cup to his father and the water heals the mortal wound.

Indy Goofed

- ▶ Although this scene adds excitement, it is somewhat embarrassing that such a distinguished professor as Dr. Indiana Jones would overlook his dominant strategy.
- ▶ He should have given the water to his father without testing it first.
 - ▶ If Indiana has chosen the right cup, his father is still saved.
 - ▶ If Indiana has chosen the wrong cup, then his father dies but Indiana is spared.
- ▶ Testing the cup before giving it to his father doesn't help, since if Indiana has made the wrong choice, there is no second chance – Indiana dies from the water and his father dies from the wound.

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Thus, everyone knows that nobody will play strictly dominated strategies in the smaller game (and such strategies may indeed exist).

Because it is common knowledge that all players will perform this kind of reasoning again, the process can continue until no more strictly dominated strategies can be eliminated.

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \dots$ of strategy sets of player i .
(Denote by G_{DS}^k the game obtained from G by restricting to $D_i^k, i \in N$.)

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Remark: If all S_i are *finite*, then in 2. we may remove only some of the strictly dominated strategies (not necessarily all). The result is *not* affected by the order of elimination since strictly dominated strategies remain strictly dominated even after removing some other strictly dominated strategies.

In the Prisoner's dilemma:

	<i>C</i>	<i>S</i>
<i>C</i>	-5, -5	0, -20
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all strategies survive all rounds (i.e. IESDS \equiv anything may happen, sorry)

A Bit More Interesting Example

	<i>L</i>	<i>C</i>	<i>R</i>
<i>L</i>	4,3	5,1	6,2
<i>C</i>	2,1	8,4	3,6
<i>R</i>	3,0	9,6	2,8

IESDS on greenboard!

Political Science Example

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- ▶ $S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (political and ideological spectrum)

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- ▶ 10 voters belong to each position
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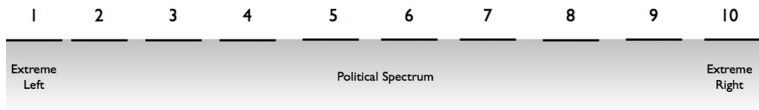
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- ▶ Payoff: The number of voters for the candidate; each candidate (selfishly) strives to maximize this number

Political Science Example



Candidate A

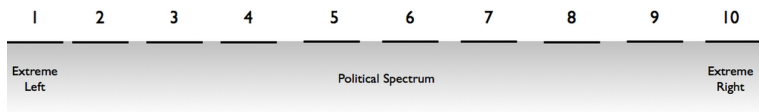


Candidates must choose to position themselves at one of the ten ideological locations. Voters are evenly distributed along the ideological spectrum, i.e. 10% at each location.



Candidate B

Political Science Example



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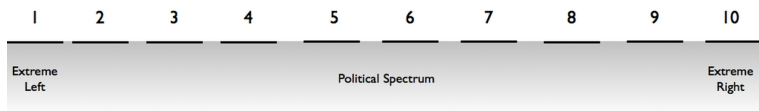
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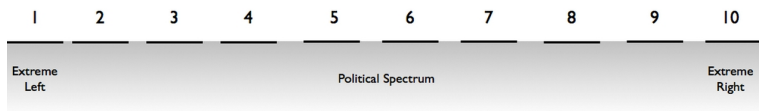
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Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

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Let us formalize this type of reasoning...

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A strategy $s_i \in S_i$ of player i is a *best response* to a belief $s_{-i} \in S_{-i}$ if

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A rational player never plays any strategy that is never best response.

Best Response vs Strict Dominance

Proposition 1

If s_i is strictly dominated for player i , then it is never best response.

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The opposite does not have to be true in pure strategies:

	X	Y
A	1, 1	1, 1
B	2, 1	0, 1
C	0, 1	2, 1

Here A is never best response but is strictly dominated neither by B, nor by C.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \dots$ of strategy sets of player i .
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(Warning: For some reasons, rationalizable strategies are almost always defined using mixed strategies!)

Rationalizability Examples

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all strategies are rationalizable.

Cournot Duopoly

$$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$$

▶ $N = \{1, 2\}$

▶ $S_i = [0, \infty)$

▶ $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$

$u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

What is a best response of player 1 to a given q_2 ?

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Since $q_2 \geq 0$, we obtain that q_1 is never best response iff $q_1 > \theta/2$.

Similarly q_2 is never best response iff $q_2 > \theta/2$.

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$$u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

What is a best response of player 1 to a given q_2 ?

Solve $\frac{\delta u_1}{\delta q_1} = \theta - 2q_1 - q_2 = 0$, which gives that $q_1 = (\theta - q_2)/2$ is the only best response of player 1 to q_2 .

Similarly, $q_2 = (\theta - q_1)/2$ is the only best response of player 2 to q_1 .

Since $q_2 \geq 0$, we obtain that q_1 is never best response iff $q_1 > \theta/2$.

Similarly q_2 is never best response iff $q_2 > \theta/2$.

Thus $R_1^1 = R_2^1 = [0, \theta/2]$.

Cournot Duopoly

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Now, in G_{Rat}^1 , we still have that $q_1 = (\theta - q_2)/2$ is the best response to q_2 , and $q_2 = (\theta - q_1)/2$ the best resp. to q_1

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Thus $R_1^2 = R_2^2 = [\theta/4, \theta/2]$.

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Cournot Duopoly (cont.)

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In general, after $2k$ iterations we have $R_i^{2k} = R_i^{2k} = [\ell_k, r_k]$ where

▶ $r_k = (\theta - \ell_{k-1})/2$ for $k \geq 1$

▶ $\ell_k = (\theta - r_k)/2$ for $k \geq 1$ and $\ell_0 = 0$

Cournot Duopoly (cont.)

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Solving the recurrence we obtain

$$\blacktriangleright \ell_k = \theta/3 - \left(\frac{1}{4}\right)^k \theta/3$$

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Hence, $\lim_{k \rightarrow \infty} \ell_k = \lim_{k \rightarrow \infty} r_k = \theta/3$ and thus $(\theta/3, \theta/3)$ is the only rationalizable equilibrium.

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Are $q_i = \theta/3$ the best outcomes possible?

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Are $q_i = \theta/3$ the best outcomes possible? NO!

$$u_1(\theta/3, \theta/3) = u_2(\theta/3, \theta/3) = \theta^2/9$$

but

$$u_1(\theta/4, \theta/4) = u_2(\theta/4, \theta/4) = \theta^2/8$$

IESDS vs Rationalizability in Pure Strategies

Theorem 14

Assume that S is finite. Then for all k we have that $R_i^k \subseteq D_i^k$. That is, in particular, all rationalizable strategies survive IESDS.

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Then $s'_i \in G_{Rat}^{k+1}$ since s'_i is *not* eliminated from G_{Rat}^k .

However, since s_i is a best response to s_{-i} in G_{Rat}^{k+1} , we get $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

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By induction hypothesis, s_j is a best response to s_{-j} in G and the claim has been proved.

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Keep in mind: If s_i is a best response to s_{-i} in G_{Rat}^k , then s_i is a best response to s_{-i} in G .

Now we prove $R_i^k \subseteq D_i^k$ for all players i by induction on k .

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By the claim, s_i is a best response to s_{-i} in G as well!

By induction hypothesis, $s_i \in R_i^{k+1} \subseteq R_i^k \subseteq D_i^k$ and $s_{-i} \in R_{-i}^k \subseteq D_{-i}^k$.

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However, then s_i is a best response to s_{-i} in G_{DS}^k .

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Thus s_i is not strictly dominated in G_{DS}^k and $s_i \in D_i^{k+1}$. □

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- ▶ Strictly dominant strategy equilibria often do not exist
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Here all strategies are equally reasonable according to the above concepts.

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But are all strategy profiles really equally reasonable?

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By Claim 3, each player plays a best response to his beliefs.

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(O, O) can be obtained as a profile where each player plays the best response to his belief and the **beliefs are correct.**

Nash Equilibrium

Nash equilibrium can be defined as a set of beliefs (one for each player) and a strategy profile in which every player plays a best response to his belief and each strategy of each player is consistent with beliefs of his opponents.

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A usual definition is following:

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A pure-strategy profile $s^* = (s_1^*, \dots, s_n^*) \in S$ is a (pure) Nash equilibrium if s_i^* is a best response to s_{-i}^* for each $i \in N$, that is

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Note that this definition is equivalent to the previous one in the sense that s_{-i}^* may be considered as the (consistent) belief of player i to which he plays a best response s_i^*

Nash Equilibria Examples

In the Prisoner's dilemma:

	<i>C</i>	<i>S</i>
<i>C</i>	-5, -5	0, -20
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only (O, O) and (F, F) are Nash equilibria.

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In the Prisoner's dilemma:

	<i>C</i>	<i>S</i>
<i>C</i>	-5, -5	0, -20
<i>S</i>	-20, 0	-1, -1

(C, C) is the only Nash equilibrium.

In the Battle of Sexes:

	<i>O</i>	<i>F</i>
<i>O</i>	2, 1	0, 0
<i>F</i>	0, 0	1, 2

only (O, O) and (F, F) are Nash equilibria.

In Cournot Duopoly, $(\theta/3, \theta/3)$ is the only Nash equilibrium.

(Best response relations: $q_1 = (\theta - q_2)/2$ and $q_2 = (\theta - q_1)/2$ are both satisfied only by $q_1 = q_2 = \theta/3$)

Example: Stag Hunt

Story:

- ▶ Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt
 - ▶ stag (S) = a large tasty meal
 - ▶ hare (H) = also tasty but small



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If each player believes that the other one will go for hare, then (H, H) is a reasonable outcome \Rightarrow a society of individualists who do not cooperate at all.

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If each player believes that the other will cooperate, then this anticipation is self-fulfilling and results in what can be called a cooperative society.

This is supposed to explain that in real world there are societies that have similar endowments, access to technology and physical environment but have very different achievements, all because of self-fulfilling beliefs (or *norms* of behavior).

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Minimum secured by playing S is 0 as opposed to 3 by playing H
(We will get to this *minimax* principle later)

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So it seems to be rational to expect (H, H) (?)

Nash Equilibria vs Previous Concepts

Theorem 16

1. *If s^* is a strictly dominant strategy equilibrium, then it is the unique Nash equilibrium.*
2. *Each Nash equilibrium is rationalizable and survives IESDS.*
3. *If S is finite, neither rationalizability, nor IESDS creates new Nash equilibria.*

Proof: Homework!

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Proof: Homework!

Corollary 17

Assume that S is finite. If rationalizability or IESDS result in a unique strategy profile, then this profile is a Nash equilibrium.

Interpretations of Nash Equilibria

Except the two definitions, usual interpretations are following:

- ▶ When the goal is to give advice to all of the players in a game (i.e., to advise each player what strategy to choose), any advice that was not an equilibrium would have the unsettling property that there would always be some player for whom the advice was bad, in the sense that, if all other players followed the parts of the advice directed to them, it would be better for some player to do differently than he was advised. If the advice is an equilibrium, however, this will not be the case, because the advice to each player is the best response to the advice given to the other players.

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- ▶ When the goal is prediction rather than prescription, a Nash equilibrium can also be interpreted as a potential stable point of a dynamic adjustment process in which individuals adjust their behavior to that of the other players in the game, searching for strategy choices that will give them better results.

Static Games of Complete Information

Mixed Strategies

Let's Mix It

As pointed out before, neither of the solution concepts has to exist in pure strategies

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Example: Rock-Paper-sCissors

	<i>R</i>	<i>P</i>	<i>C</i>
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How to solve this?

Let the players randomize their choice of pure strategies

Probability Distributions

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Let A be a finite set. A *probability distribution over A* is a function $\sigma : A \rightarrow [0, 1]$ such that $\sum_{a \in A} \sigma(a) = 1$.

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Example 19

Consider $A = \{a, b, c\}$ and a function $\sigma : A \rightarrow [0, 1]$ such that $\sigma(a) = \frac{1}{4}$, $\sigma(b) = \frac{3}{4}$, and $\sigma(c) = 0$. Then $\sigma \in \Delta(A)$.

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For example, in rock-paper-scissors, the pure strategy R corresponds

to σ_i which satisfies $\sigma_i(X) = \begin{cases} 1 & X = R \\ 0 & \text{otherwise} \end{cases}$

Mixed Strategy Profiles

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Thus for $s = (s_1, s_2) \in S = S_1 \times S_2$ we have that

$$\sigma(s) := \sigma_1(s_1) \cdot \sigma_2(s_2)$$

is the probability that the players randomly select the pure strategy profile s according to the mixed strategy profile σ .

(We abuse notation a bit here: σ denotes two things, a vector of mixed strategies as well as a probability distribution on S)

Mixed Strategies – Example

	<i>R</i>	<i>P</i>	<i>C</i>
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An example of a mixed strategy σ_1 : $\sigma_1(R) = \frac{1}{2}$, $\sigma_1(P) = \frac{1}{3}$, $\sigma_1(C) = \frac{1}{6}$.

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Consider a mixed strategy profile (σ_1, σ_2) where $\sigma_1 = (\frac{1}{2}(R), \frac{1}{3}(P), \frac{1}{6}(C))$ and $\sigma_2 = (\frac{1}{3}(R), \frac{2}{3}(P), 0(C))$.

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Then the probability $\sigma(R, P)$ that the pure strategy profile (R, P) will be played by players playing the mixed profile (σ_1, σ_2) is

$$\sigma_1(R) \cdot \sigma_2(P) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Expected Payoff

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Definition 21

The *expected payoff* of player i under a mixed strategy profile $\sigma \in \Sigma$ is

$$u_i(\sigma) := \sum_{s \in S} \sigma(s) u_i(s) \quad \left(= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdot u_i(s_1, s_2) \right)$$

I.e., it is the "weighted average" of what player i wins under each pure strategy profile s , weighted by the probability of that profile.

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Assumption: Every rational player strives to maximize his own expected payoff.

(This assumption is not always completely convincing ...)

Expected Payoff – Example

Matching Pennies:

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Each player secretly turns a penny to heads or tails, and then they reveal their choices simultaneously. If the pennies match, player 1 (row) wins, if they do not match, player 2 (column) wins.

Consider $\sigma_1 = (\frac{1}{3}(H), \frac{2}{3}(T))$ and $\sigma_2 = (\frac{1}{4}(H), \frac{3}{4}(T))$

$$\begin{aligned}u_1(\sigma_1, \sigma_2) &= \sum_{(X,Y) \in \{H,T\}^2} \sigma_1(X)\sigma_2(Y)u_1(X, Y) \\ &= \frac{1}{3} \frac{1}{4} 1 + \frac{1}{3} \frac{3}{4} (-1) + \frac{2}{3} \frac{1}{4} (-1) + \frac{2}{3} \frac{3}{4} 1 = \frac{1}{6}\end{aligned}$$

$$\begin{aligned}u_2(\sigma_1, \sigma_2) &= \sum_{(X,Y) \in \{H,T\}^2} \sigma_1(X)\sigma_2(Y)u_2(X, Y) \\ &= \frac{1}{3} \frac{1}{4} (-1) + \frac{1}{3} \frac{3}{4} 1 + \frac{2}{3} \frac{1}{4} 1 + \frac{2}{3} \frac{3}{4} (-1) = -\frac{1}{6}\end{aligned}$$

Solution Concepts

We revisit the following solution concepts in mixed strategies:

- ▶ strict dominant strategy equilibrium
- ▶ IESDS equilibrium
- ▶ rationalizable equilibria
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mixed strategy.

In order to deal with efficiency issues we assume that the size of the game G is defined by $|G| := |N| + \sum_{i \in N} |S_i| + \sum_{i \in N} |u_i|$ where $|u_i| = \sum_{s \in S} |u_i(s)|$ and $|u_i(s)|$ is the length of a binary encoding of $u_i(s)$ (we assume that rational numbers are encoded as quotients of two binary integers)

Note that, in particular, $|G| > |S|$.

Strict Dominance in Mixed Strategies

Definition 22

Let $\sigma_1, \sigma'_1 \in \Sigma_1$ be (mixed) strategies of player 1. Then σ'_1 is *strictly dominated* by σ_1 (write $\sigma'_1 < \sigma_1$) if

$$u_1(\sigma_1, s_2) > u_1(\sigma'_1, s_2) \quad \text{for all } s_2 \in S_2$$

(Symmetrically for player 2.)

Comment: The above condition is equivalent to

$$u_1(\sigma_1, \sigma_2) > u_1(\sigma'_1, \sigma_2) \quad \text{for all strategies } \sigma_2 \in \Sigma_2$$

Strict Dominance in Mixed Strategies

Example 23

	X	Y
A	3	0
B	0	3
C	1	1

Is there a strictly dominated strategy?

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Question: Is there a game with at least one strictly dominated strategy but without strictly dominated *pure* strategies?

Strictly Dominant Strategy Equilibrium

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$\sigma_i \in \Sigma_i$ is *strictly dominant* if every other mixed strategy of player i is strictly dominated by σ_i .

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A strategy profile $\sigma \in \Sigma$ is a *strictly dominant strategy equilibrium* if $\sigma_i \in \Sigma_i$ is strictly dominant for each $i \in N$.

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Proposition 2

If the strictly dominant strategy equilibrium exists, it is unique; all its strategies are pure, and rational players will play it.

Proof.

Homework. □

To compute the strictly dominant strategy equilibrium, it is sufficient to consider only pure strategies.

IESDS in Mixed Strategies

Define a sequence $D_i^0, D_i^1, D_i^2, \dots$ of strategy sets of player i .
(Denote by G_{DS}^k the game obtained from G by restricting the pure strategy sets to $D_i^k, i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are *not* strictly dominated in G_{DS}^k by *mixed strategies*.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ *survives IESDS* if $s_i \in D_i^k$ for all $k = 0, 1, 2, \dots$

Definition 26

A strategy profile $s = (s_1, s_2) \in S$ is an *IESDS equilibrium* if both s_1 and s_2 survive IESDS.

Each D_i^{k+1} can be computed in polynomial time using *linear programming*.

IESDS in Mixed Strategie – Example

	X	Y
A	3	0
B	0	3
C	1	1

Let us have a look at the first iteration of IESDS.

IESDS in Mixed Strategie – Example

	X	Y
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C	1	1

Let us have a look at the first iteration of IESDS.

Observe that A, B are not strictly dominated by any mixed strategy.

IESDS in Mixed Strategie – Example

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Let us construct a set of constraints on mixed strategies (possibly) strictly dominating C :

$$3x_A + 0x_B + x_C > 1$$

Row's payoff against X

$$0x_A + 3x_B + x_C > 1$$

Row's payoff against Y

$$x_A, x_B, x_C \geq 0$$

$$x_A + x_B + x_C = 1$$

x 's must make a distribution

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How to solve this?

Intermezzo: Linear Programming

Linear programming is a technique for optimization of a linear objective function, subject to linear (non-strict) inequality constraints.

Formally, a linear program in so called *canonical form* looks like this:

$$\text{maximize } \sum_{j=1}^m c_j x_j \quad (\text{objective function})$$

$$\text{subject to } \sum_{j=1}^m a_{ij} x_j \leq b_i \quad 1 \leq i \leq n \quad (\text{constraints})$$

$$x_j \geq 0 \quad 1 \leq j \leq m$$

Here a_{ij} , b_k and c_j are real numbers and x_j 's are real variables.

A *feasible solution* is an assignment of real numbers to the variables x_j , $1 \leq j \leq m$, so that the *constraints* are satisfied.

An *optimal solution* is a feasible solution which maximizes the *objective function* $\sum_{j=1}^m c_j x_j$.

Intermezzo: Complexity of Linear Programming

We assume that coefficients a_{ij} , b_k and c_j are encoded in binary (more precisely, as fractions of two integers encoded in binary).

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There is an algorithm which for any linear program computes an optimal solution in polynomial time.

The algorithm uses so called ellipsoid method.

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For more info see

http://en.wikipedia.org/wiki/Linear_programming#Solvers_and_scripting_.28programming.29_languages

IESDS in Mixed Strategie – Example

	X	Y
A	3	0
B	0	3
C	1	1

The linear program for deciding whether C is strictly dominated: The program maximizes y under the following constraints:

$$3x_A + 0x_B + x_C \geq 1 + y$$

Row's payoff against X

$$0x_A + 3x_B + x_C \geq 1 + y$$

Row's payoff against Y

$$x_A, x_B, x_C \geq 0$$

$$x_A + x_B + x_C = 1$$

x's must make a distribution

$$y \geq 0$$

Here y just implements the strict inequality using \geq , we look for a solution with $y > 0$.

The maximum $y = \frac{1}{2}$ is attained at $x_A = \frac{1}{2}$ and $x_B = \frac{1}{2}$.

Note that in step 2 it is not sufficient to consider pure strategies.
Consider the following zero sum game:

	X	Y
A	3	0
B	0	3
C	1	1

C is strictly dominated by $(\sigma_1(A), \sigma_1(B), \sigma_1(C)) = (\frac{1}{2}, \frac{1}{2}, 0)$ but no strategy is strictly dominated in pure strategies.

Best Response in Mixed Strategies

Definition 28

A *(mixed) belief* of player 1 is a mixed strategy σ_2 of player 2 (and vice versa).

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Definition 29

$\sigma_1 \in \Sigma_1$ is a *best response* to a belief $\sigma_2 \in \Sigma_2$ if

$$u_1(\sigma_1, \sigma_2) \geq u_1(\mathbf{s}_1, \sigma_2) \quad \text{for all } \mathbf{s}_1 \in \mathbf{S}_1$$

Denote by $BR_1(\sigma_2)$ the set of all best responses of player 1. (Symmetrically for player 2.)

Comment: The above condition is equivalent to

$$u_1(\sigma_1, \sigma_2) \geq u_1(\sigma'_1, \sigma_2) \quad \text{for all } \sigma'_1 \in \Sigma_1$$

Best Response – Example

Consider a game with the following payoffs of player 1:

	X	Y
A	2	0
B	0	2
C	1	1

- ▶ Player 1 (row) plays $\sigma_1 = (a(A), b(B), c(C))$.
- ▶ Player 2 (column) plays $(q(X), (1 - q)(Y))$ (we write just q).

Compute $BR_1(q)$.

Rationalizability in Mixed Strategies (Two Players)

Assumption: *A rational player 1 with a belief σ_2 always plays a best response to σ_2 (the same for player 2).*

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Definition 30

A pure strategy $s_1 \in S_1$ of player 1 is *never best response* if it is not a best response to any belief σ_2 (similarly for player 2).

No rational player plays a strategy that is never best response.

Rationalizability in Mixed Strategies (Two Players)

Define a sequence $R_i^0, R_i^1, R_i^2, \dots$ of strategy sets of player i .

(Denote by G_{Rat}^k the game obtained from G by restricting the pure strategy sets to $R_i^k, i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are *best responses to some (mixed) beliefs* in G_{Rat}^k .
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ is *rationalizable* if $s_i \in R_i^k$ for all $k = 0, 1, 2, \dots$

Definition 31

A strategy profile $s = (s_1, s_2) \in S$ is a *rationalizable equilibrium* if both s_1 and s_2 are rationalizable.

Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
B	0	3
C	1	1

What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
B	0	3
C	1	1

What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

Observation: The set of strictly dominated pure strategies coincides with the set of pure never best responses!

Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
B	0	3
C	1	1

What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

Observation: The set of strictly dominated pure strategies coincides with the set of pure never best responses!

... and this holds in general for two player games:

Theorem 32

A pure strategy s_1 of player 1 is never best response to any belief σ_2 **iff** s_1 is strictly dominated by a strategy $\sigma_1 \in \Sigma_1$ (similarly for player 2).

It follows that a strategy of S_i survives IESDS **iff** it is rationalizable.

Mixed Nash Equilibrium

Definition 33

A mixed-strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ is a (mixed) Nash equilibrium if σ_1^* is a best response to σ_2^* and σ_2^* is a best response to σ_1^* . That is

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(\mathbf{s}_1, \sigma_2^*) \quad \text{for all } \mathbf{s}_1 \in \mathbf{S}_1$$

$$u_2(\sigma_1^*, \sigma_2^*) \geq u_2(\sigma_1^*, \mathbf{s}_2) \quad \text{for all } \mathbf{s}_2 \in \mathbf{S}_2$$

The above condition is equivalent to

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Theorem 34 (Nash 1950)

Every finite game in strategic form has a Nash equilibrium.

This is THE fundamental theorem of game theory.

Example: Matching Pennies

	<i>H</i>	<i>T</i>
<i>H</i>	1, -1	-1, 1
<i>T</i>	-1, 1	1, -1

Player 1 (row) plays $(p(H), (1 - p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1 - q)(T))$ (we write q).

Compute all Nash equilibria.

Example: Matching Pennies

	<i>H</i>	<i>T</i>
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Compute all Nash equilibria.

What are the expected payoffs of playing pure strategies for player 1?

$$u_1(H, q) = 2q - 1 \text{ and } u_1(T, q) = 1 - 2q$$

Example: Matching Pennies

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$$u_1(p, q) = pu_1(H, q) + (1 - p)u_1(T, q) = p(2q - 1) + (1 - p)(1 - 2q).$$

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$$u_1(p, q) = pu_1(H, q) + (1 - p)u_1(T, q) = p(2q - 1) + (1 - p)(1 - 2q).$$

We obtain the best response correspondence BR_1 :

$$BR_1(q) = \begin{cases} T & \text{if } q < \frac{1}{2} \\ p \in [0, 1] & \text{if } q = \frac{1}{2} \\ H & \text{if } q > \frac{1}{2} \end{cases}$$

Example: Matching Pennies

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<i>H</i>	1, -1	-1, 1
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Compute all Nash equilibria.

Similarly for player 2 :

$$u_2(p, H) = 1 - 2p \text{ and } u_2(p, T) = 2p - 1$$

Example: Matching Pennies

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Example: Matching Pennies

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We obtain best-response relation BR_2 :

$$BR_2(p) = \begin{cases} H & \text{if } p < \frac{1}{2} \\ q \in [0, 1] & \text{if } p = \frac{1}{2} \\ T & \text{if } p > \frac{1}{2} \end{cases}$$

Example: Matching Pennies

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The only "intersection" of BR_1 and BR_2 is the only Nash equilibrium

$$\sigma_1 = \sigma_2 = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Support Enumeration

Computing Mixed Nash Equilibria

Lemma 35

Every Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ satisfies

- ▶ $u_1(s_1, \sigma_2^*) = u_1(\sigma^*)$ for $s_1 \in \text{supp}(\sigma_1^*)$
- ▶ $u_2(\sigma_1^*, s_2) = u_2(\sigma^*)$ for $s_2 \in \text{supp}(\sigma_2^*)$

Computing Mixed Nash Equilibria

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Proof. W.l.o.g. consider only the player 1 and assume that σ^* is a Nash equilibrium.

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Proof. W.l.o.g. consider only the player 1 and assume that σ^* is a Nash equilibrium.

The latter assumption implies $u_1(s_1, \sigma_2^*) \leq u_1(\sigma^*)$ for all $s_1 \in S_1$.

Computing Mixed Nash Equilibria

Lemma 35

Every Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ satisfies

- ▶ $u_1(s_1, \sigma_2^*) = u_1(\sigma^*)$ for $s_1 \in \text{supp}(\sigma_1^*)$
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Proof. W.l.o.g. consider only the player 1 and assume that σ^* is a Nash equilibrium.

The latter assumption implies $u_1(s_1, \sigma_2^*) \leq u_1(\sigma^*)$ for all $s_1 \in S_1$.

Now, if there exists $s'_1 \in \text{supp}(\sigma_1^*) \subseteq S_1$ satisfying $u_1(s'_1, \sigma_2^*) < u_1(\sigma^*)$, then because $\sigma_1^*(s'_1) > 0$ we have

$$u_1(\sigma^*) = \sum_{s_1 \in S_1} \sigma_1^*(s_1) u_1(s_1, \sigma_2^*) < \sum_{s_1 \in S_1} \sigma_1^*(s_1) u_1(\sigma^*) = u_1(\sigma^*)$$

A contradiction.

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Every Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ satisfies

- ▶ $u_1(s_1, \sigma_2^*) = u_1(\sigma^*)$ for $s_1 \in \text{supp}(\sigma_1^*)$
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Proof. W.l.o.g. consider only the player 1 and assume that σ^* is a Nash equilibrium.

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Now, if there exists $s'_1 \in \text{supp}(\sigma_1^*) \subseteq S_1$ satisfying $u_1(s'_1, \sigma_2^*) < u_1(\sigma^*)$, then because $\sigma_1^*(s'_1) > 0$ we have

$$u_1(\sigma^*) = \sum_{s_1 \in S_1} \sigma_1^*(s_1) u_1(s_1, \sigma_2^*) < \sum_{s_1 \in S_1} \sigma_1^*(s_1) u_1(\sigma^*) = u_1(\sigma^*)$$

A contradiction.

Thus $u_1(s_1, \sigma_2^*) = u_1(\sigma^*)$ for all $s_1 \in \text{supp}(\sigma_1^*)$.

Example: Matching Pennies

	<i>H</i>	<i>T</i>
<i>H</i>	1, -1	-1, 1
<i>T</i>	-1, 1	1, -1

Player 1 (row) plays $(p(H), (1 - p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1 - q)(T))$ (we write q).

Compute all Nash equilibria.

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Compute all Nash equilibria.

There are no pure strategy equilibria.

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Compute all Nash equilibria.

There are no pure strategy equilibria.

There are no equilibria where only player 1 randomizes:

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	<i>H</i>	<i>T</i>
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Player 1 (row) plays $(p(H), (1 - p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1 - q)(T))$ (we write q).

Compute all Nash equilibria.

There are no pure strategy equilibria.

There are no equilibria where only player 1 randomizes:

Indeed, assume that (p, H) is such an equilibrium. Then by Lemma 35,

$$1 = u_1(H, H) = u_1(T, H) = -1$$

a contradiction. Also, (p, T) cannot be an equilibrium.

Similarly, there is no NE where only player 2 randomizes.

Example: Matching Pennies

	<i>H</i>	<i>T</i>
<i>H</i>	1, -1	-1, 1
<i>T</i>	-1, 1	1, -1

Player 1 (row) plays $(p(H), (1 - p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1 - q)(T))$ (we write q).

Compute all Nash equilibria.

Assume that both players randomize, i.e., $p, q \in (0, 1)$.

Example: Matching Pennies

	<i>H</i>	<i>T</i>
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Compute all Nash equilibria.

Assume that both players randomize, i.e., $p, q \in (0, 1)$.

The expected payoffs of playing pure strategies for player 1:

$$u_1(H, q) = 2q - 1 \text{ and } u_1(T, q) = 1 - 2q$$

Similarly for player 2 :

$$u_2(p, H) = 1 - 2p \text{ and } u_2(p, T) = 2p - 1$$

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Similarly for player 2 :

$$u_2(p, H) = 1 - 2p \text{ and } u_2(p, T) = 2p - 1$$

By Lemma 35, such Nash equilibria must satisfy:

$$2q - 1 = 1 - 2q \quad \text{and} \quad 1 - 2p = 2p - 1$$

That is $p = q = \frac{1}{2}$ is the only Nash equilibrium.

Example: Battle of Sexes

	O	F
O	2,1	0,0
F	0,0	1,2

Player 1 (row) plays $(p(O), (1 - p)(F))$ (we write just p) and player 2 (column) plays $(q(O), (1 - q)(F))$ (we write q).

Compute all Nash equilibria.

Example: Battle of Sexes

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where $p, q \in (0, 1)$.

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where $p, q \in (0, 1)$.

By Lemma 35, such Nash equilibria must satisfy:

$$2q = 1 - q \quad \text{and} \quad p = 2(1 - p)$$

This holds only for $q = \frac{1}{3}$ and $p = \frac{2}{3}$.

An Algorithm?

What did we do in the previous examples?

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Whenever one of the *supports* was non-singleton, we reduced computation of Nash equilibria to *linear equations*.

Computing Mixed Nash Equilibria

Lemma 36

Let $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ be a mixed profile. Assume that there exist $w_1, w_2 \in \mathbb{R}$ such that

- ▶ $u_1(s_1, \sigma_2^*) = w_1$ for $s_1 \in \text{supp}(\sigma_1^*)$
- ▶ $u_1(s_1, \sigma_2^*) \leq w_1$ for $s_1 \notin \text{supp}(\sigma_1^*)$
- ▶ $u_2(\sigma_1^*, s_2) = w_2$ for $s_2 \in \text{supp}(\sigma_2^*)$
- ▶ $u_2(\sigma_1^*, s_2) \leq w_2$ for $s_2 \notin \text{supp}(\sigma_2^*)$

Then $u_1(\sigma^*) = w_1$ and $u_2(\sigma^*) = w_2$, and σ^* is a Nash equilibrium.

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Proof. Consider just the player 1 (for pl. 2 similarly):

$$\begin{aligned} u_1(\sigma^*) &= \sum_{s_1 \in S_1} \sigma^*(s_1) u_1(s_1, \sigma_2^*) = \sum_{s_1 \in \text{supp}(\sigma_1^*)} \sigma^*(s_1) u_1(s_1, \sigma_2^*) \\ &= \sum_{s_1 \in \text{supp}(\sigma_1^*)} \sigma^*(s_1) w_1 = w_1 \sum_{s_1 \in \text{supp}(\sigma_1^*)} \sigma^*(s_1) = w_1 \end{aligned}$$

Now the fact that σ^* is a Nash equilibrium follows from the definition.

How to Compute Mixed Nash Equilibria?

Every Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*)$ can be computed by finding appropriate w_1, w_2 so that

- ▶ $u_1(s_1, \sigma_2^*) = w_1$ for $s_1 \in \text{supp}(\sigma_1^*)$
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Indeed,

- ▶ by Lemma 36, all σ^* and w_1, w_2 satisfying the above inequalities give a Nash equilibrium σ^* with $u_1(\sigma^*) = w_1$ and $u_2(\sigma^*) = w_2$,
- ▶ by Lemma 35, for every Nash equilibrium σ^* choosing $w_1 = u_1(\sigma^*)$ and $w_2 = u_2(\sigma^*)$ satisfies the above inequalities.

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Suppose that we somehow know the supports $\text{supp}(\sigma_1^*), \text{supp}(\sigma_2^*)$ for some Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*)$ (which itself is unknown to us).

We may consider all $\sigma_i^*(s_i)$'s and both w_1, w_2 's as variables and use the above conditions to design a system of inequalities capturing Nash equilibria with the given support sets $\text{supp}(\sigma_1^*), \text{supp}(\sigma_2^*)$.

Support Enumeration

To simplify notation, assume that for every i we have $S_i = \{1, \dots, m_i\}$.
Then $\sigma_i(j)$ is the probability of the pure strategy j in the mixed strategy σ_i .

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$\sigma_1(1), \dots, \sigma_1(m_1), \sigma_2(1), \dots, \sigma_2(m_2), w_1, w_2$:

1. For all $k \in supp_1$ and all $\ell \in supp_2$:

$$\sum_{\ell' \in S_2} \sigma_2(\ell') u_1(k, \ell') = w_1 \quad \sum_{k' \in S_1} \sigma_1(k') u_2(k', \ell) = w_2$$

2. For all $k \notin supp_1$ and all $\ell \notin supp_2$:

$$\sum_{\ell' \in S_2} \sigma_2(\ell') u_1(k, \ell') \leq w_1 \quad \sum_{k' \in S_1} \sigma_1(k') u_2(k', \ell) \leq w_2$$

3. For all $i \in \{1, 2\}$: $\sigma_i(1) + \dots + \sigma_i(m_i) = 1$.
4. For all $i \in \{1, 2\}$ and all $k \in supp_i$: $\sigma_i(k) \geq 0$.
5. For all $i \in \{1, 2\}$ and all $k \notin supp_i$: $\sigma_i(k) = 0$.

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How to find $supp_1$ and $supp_2$? ... Just guess!

Input: A two-player strategic-form game G with strategy sets $S_1 = \{1, \dots, m_1\}$ and $S_2 = \{1, \dots, m_2\}$ and rational payoffs u_1, u_2 .

Output: A Nash equilibrium σ^* .

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Algorithm: For all possible $supp_1 \subseteq S_1$ and $supp_2 \subseteq S_2$:

- ▶ Check if the corresponding system of linear constraints (from the previous slide) has a feasible solution σ^*, w_1^*, w_2^* .
- ▶ If so, STOP: the feasible solution σ^* is a Nash equilibrium satisfying $u_i(\sigma^*) = w_i^*$.

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Question: How many possible subsets $supp_1, supp_2$ are there to try?

Answer: $2^{(m_1+m_2)}$

So, unfortunately, the algorithm requires worst-case exponential time.

Remarks on Support Enumeration

- ▶ The algorithm combined with Theorem 34 and properties of linear programming imply that every finite two-player game has a rational Nash equilibrium (furthermore, the rational numbers have polynomial representation in binary).

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(There are algorithms for computing (a finite representation of) a set of all feasible solutions of a given linear constraint system.)
- ▶ The algorithm can be used to compute "good" equilibria.

For example, to find a Nash equilibrium maximizing the sum of all expected payoffs (the "social welfare") it suffices to solve the system of constraints while maximizing $w_1 + w_2$. More precisely, the algorithm can be modified as follows:

- ▶ Initialize $W := -\infty$ (W stores the current maximum welfare)
- ▶ For all possible $supp_1 \subseteq S_1$ and $supp_2 \subseteq S_2$:
 - ▶ Find the maximum value $\max(w_1 + w_2)$ of $w_1 + w_2$ so that the constraints are satisfiable (using linear programming).
 - ▶ Put $W := \max\{W, \max(w_1 + w_2)\}$.
- ▶ Return W .

Remarks on Support Enumeration (Cont.)

Similar trick works for any notion of "good" NE that can be expressed using a linear objective function and (additional) linear constraints in variables $\sigma_i(j)$ and w_j .

(e.g., maximize payoff of player 1, minimize payoff of player 2 and keep probability of playing the strategy 1 below 1/2, etc.)

Complexity Results – (Two Players)

Theorem 37

Given a two-player game in strategic form, a mixed Nash equilibrium can be computed in exponential time.

Theorem 38

All the following problems are NP-complete: Given a two-player game in strategic form, does it have

- 1. a NE in which player 1 has utility at least a given amount v ?*
- 2. a NE in which the sum of expected payoffs of the two players is at least a given amount v ?*
- 3. a NE with a support of size greater than a given number?*
- 4. a NE whose support contains a given strategy s ?*
- 5. a NE whose support does not contain a given strategy s ?*
- 6.*

NP-hardness can be proved using reduction from SAT.

The Reduction (It's Short and Sweet)

Definition 4 Let ϕ be a Boolean formula in conjunctive normal form (representing a SAT instance). Let V be its set of variables (with $|V| = n$), L the set of corresponding literals (a positive and a negative one for each variable⁶), and C its set of clauses. The function $v : L \rightarrow V$ gives the variable corresponding to a literal, e.g., $v(x_1) = v(-x_1) = x_1$. We define $G_\epsilon(\phi)$ to be the following finite symmetric 2-player game in normal form. Let $\Sigma = \Sigma_1 = \Sigma_2 = L \cup V \cup C \cup \{f\}$. Let the utility functions be

- $u_1(l^1, l^2) = u_2(l^2, l^1) = n - 1$ for all $l^1, l^2 \in L$ with $l^1 \neq -l^2$;
- $u_1(l, -l) = u_2(-l, l) = n - 4$ for all $l \in L$;
- $u_1(l, x) = u_2(x, l) = n - 4$ for all $l \in L, x \in \Sigma - L - \{f\}$;
- $u_1(v, l) = u_2(l, v) = n$ for all $v \in V, l \in L$ with $v(l) \neq v$;
- $u_1(v, l) = u_2(l, v) = 0$ for all $v \in V, l \in L$ with $v(l) = v$;
- $u_1(v, x) = u_2(x, v) = n - 4$ for all $v \in V, x \in \Sigma - L - \{f\}$;
- $u_1(c, l) = u_2(l, c) = n$ for all $c \in C, l \in L$ with $l \notin c$;
- $u_1(c, l) = u_2(l, c) = 0$ for all $c \in C, l \in L$ with $l \in c$;
- $u_1(c, x) = u_2(x, c) = n - 4$ for all $c \in C, x \in \Sigma - L - \{f\}$;
- $u_1(x, f) = u_2(f, x) = 0$ for all $x \in \Sigma - \{f\}$;
- $u_1(f, f) = u_2(f, f) = \epsilon$;
- $u_1(f, x) = u_2(x, f) = n - 1$ for all $x \in \Sigma - \{f\}$.

Theorem 1 If (l_1, l_2, \dots, l_n) (where $v(l_i) = x_i$) satisfies ϕ , then there is a Nash equilibrium of $G_\epsilon(\phi)$ where both players play l_i with probability $\frac{1}{n}$, with expected utility $n - 1$ for each player. The only other Nash equilibrium is the one where both players play f , and receive expected utility ϵ each.

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As the class NP consists of decision problems, it cannot be directly used to characterize complexity of the sample equilibrium problem.

We use complexity classes of *function problems* such as FP, FNP, etc. The sample equilibrium problem belongs to the complexity class PPAD (which is a subclass of TFNP) for two-player games.

A binary relation $P(x,y)$ is in TFNP if and only if there is a deterministic polynomial time algorithm that can determine whether $P(x,y)$ holds given both x and y , and for every x , there exists a y which is at most polynomially longer than x such that $P(x,y)$ holds.

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Can we do better than FNP (i.e. exponential time)?

In what follows we show that the sample equilibrium problem can be solved in polynomial time for zero-sum two-player games.

(Using a beautiful characterization of all Nash equilibria)

Definition 39

$\sigma_1^* \in \Sigma_1$ is a *maxmin* strategy of player 1 if

$$\sigma_1^* \in \underset{\sigma_1 \in \Sigma_1}{\operatorname{argmax}} \min_{s_2 \in S_2} u_1(\sigma_1, s_2) \quad (= \underset{\sigma_1 \in \Sigma_1}{\operatorname{argmax}} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2))$$

(Intuitively, a *maxmin* strategy σ_1^* maximizes player 1's worst-case payoff in the situation where player 2 strives to cause the greatest harm to player 1.)

Similarly, $\sigma_2^* \in \Sigma_2$ is a *maxmin* strategy of player 2 if

$$\sigma_2^* \in \underset{\sigma_2 \in \Sigma_2}{\operatorname{argmax}} \min_{s_1 \in S_1} u_2(s_1, \sigma_2)$$

Which assuming zero-sum games, i.e. $u_1 = -u_2$, becomes

$$\sigma_2^* \in \underset{\sigma_2 \in \Sigma_2}{\operatorname{argmin}} \max_{s_1 \in S_1} u_1(s_1, \sigma_2) \quad (= \underset{\sigma_2 \in \Sigma_2}{\operatorname{argmin}} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2))$$

Note the same payoff function for both players!!

Theorem 40 (von Neumann)

Assume a two-player **zero-sum** game. Then

$$\max_{\sigma_1 \in \Sigma_1} \min_{s_2 \in S_2} u_1(\sigma_1, s_2) = \min_{\sigma_2 \in \Sigma_2} \max_{s \in S_1} u_1(s_1, \sigma_2)$$

Moreover, $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ is a Nash equilibrium **iff** both σ_1^* and σ_2^* are *maxmin*.

So to compute a Nash equilibrium it suffices to compute (arbitrary) maxmin strategies for both players.

Zero-Sum Two-Player Games – Computing NE

Assume $S_1 = \{1, \dots, m_1\}$ and $S_2 = \{1, \dots, m_2\}$.

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Consider a linear program with variables $\sigma_1(1), \dots, \sigma_1(m_1), v$:

maximize: v

subject to:
$$\sum_{k=1}^{m_1} \sigma_1(k) \cdot u_1(k, \ell) \geq v \quad \ell = 1, \dots, m_2$$

$$\sum_{k=1}^{m_1} \sigma_1(k) = 1$$

$$\sigma_1(k) \geq 0 \quad k = 1, \dots, m_1$$

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Lemma 41

$\sigma_1^* \in \operatorname{argmax}_{\sigma_1 \in \Sigma_1} \min_{\ell \in S_2} u_1(\sigma_1, \ell)$ **iff** assigning $\sigma_1(k) := \sigma_1^*(k)$ and $v := \min_{\ell \in S_2} u_1(\sigma_1^*, \ell)$ gives an optimal solution.

Summary:

- ▶ We have reduced computation of NE to computation of maxmin strategies for both players.
- ▶ Maxmin strategies can be computed using linear programming in polynomial time.
- ▶ That is, Nash equilibria in zero-sum two-player games can be computed in polynomial time.

Strategic-Form Games – Conclusion

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We modeled such games using *strategic-form games*.

We have considered both pure strategy setting and mixed strategy setting.

In both cases, we considered four solution concepts:

- ▶ Strictly dominant strategies
- ▶ Iterative elimination of strictly dominated strategies
- ▶ Rationalizability (i.e., iterative elimination of strategies that are never best responses)
- ▶ Nash equilibria

Strategic-Form Games – Conclusion

In pure strategy setting:

1. Strictly dominant strategy equilibrium survives IESDS, rationalizability and is the unique Nash equilibrium (if it exists)
2. In finite games, rationalizable equilibria survive IESDS, IESDS preserves the set of Nash equilibria
3. In finite games, rationalizability preserves Nash equilibria

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In mixed setting:

1. In finite two player games, IESDS and rationalizability coincide.
2. Strictly dominant strategy equilibrium survives IESDS (rationalizability) and is the unique Nash equilibrium (if it exists)
3. In finite games, IESDS (rationalizability) preserves Nash equilibria

The proofs for 2. and 3. in the mixed setting are similar to corresponding proofs in the pure setting.

- ▶ Strictly dominant strategy equilibria coincide in pure and mixed settings, and can be computed in polynomial time.

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Algorithms

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- ▶ IESDS and rationalizability can be implemented in polynomial time in the pure setting as well as in the mixed setting
In the mixed setting, linear programming is needed to implement one step of IESDS (rationalizability).
- ▶ Nash equilibria can be computed for two-player games
 - ▶ in polynomial time for zero-sum games
(using von Neumann's theorem and linear programming)
 - ▶ in exponential time using support enumeration
 - ▶ in PPAD using Lemke-Howson (omitted)

Loose Ends – Modes of Dominance

To simplify, let us consider only **pure strategies**.

Let $s_i, s'_i \in S_i$. Then s'_i is *strictly dominated* by s_i if $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

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Let $s_i, s'_i \in S_i$. Then s'_i is *weakly dominated* by s_i if $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ and there is $s'_{-i} \in S_{-i}$ such that $u_i(s_i, s'_{-i}) > u_i(s'_i, s'_{-i})$.

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Claim 4

Any pure strategy profile $s \in S$ such that each s_i is very weakly dominant is a Nash equilibrium.

The same claim can be proved in the mixed strategy setting.