IA168 Algorithmic Game Theory

Tomáš Brázdil

Organization of This Course

Sources:

- Lectures (slides, notes)
  - based on several sources
  - slides are prepared for lectures, some stuff on greenboard ($\Rightarrow$ attend the lectures)

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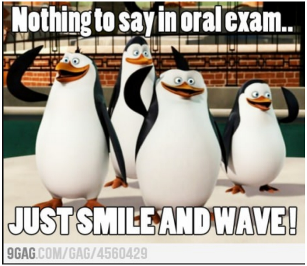
- Books:
  - Nisan/Roughgarden/Tardos/Vazirani, **Algorithmic Game Theory**, Cambridge University, 2007.
    Available online for free:
    http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf
  - Tadelis, **Game Theory: An Introduction**, Princeton University Press, 2013

(I use various resources, so please, attend the lectures)

Evaluation

- **Oral exam**
- Homework



  - 3 homework assignments

Notable features of the course

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- **Very demanding!**
- Mathematical!

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An example of an instruction email (from another course with the same system):

```
It is typically not sufficient to devote a single
afternoon to the preparation for the exam.
You have to know _everything_ (which means every
single thing) starting with the slide 42
and ending with the slide 245 with notable exceptions
of slides: 121 - 123, 137 - 140, 165, 167.
Proofs presented on the whiteboard are also mandatory.
```

Most importantly,

The previous slide is not a joke!

What is Algorithmic Game Theory?

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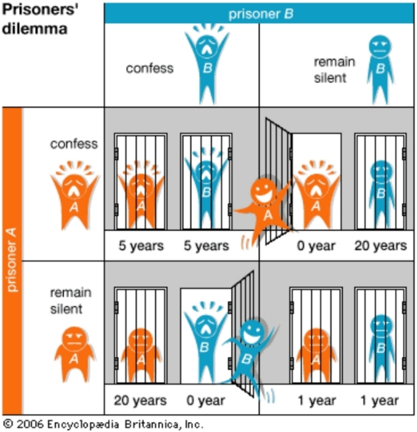
What does the "algorithmic" mean?

- It means that we are "concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."

Let's have a look at some examples ....

Prisoner's Dilemma



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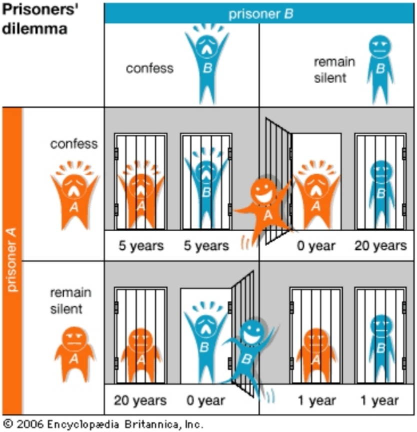

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- Each of the suspects is offered a deal: If he confesses (C) to the crime, he is free to go. The alternative is not to confess, that is remain silent (S).

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Sentence depends on the behavior of both suspects.
The problem: What would the suspects do?

Prisoner's Dilemma – Solution(?)

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

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Where is the dilemma? There is a solution (S, S) which is better for both players but needs some "central" authority to control the players.

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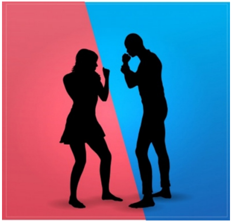
Are there always "dominant" strategies?

Nash equilibria – Battle of Sexes



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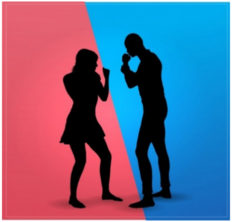

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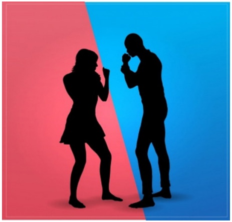

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If they cannot communicate, where should they go?

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Battle of Sexes can be modeled as a game of two players (the couple) with the following payoffs:

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(O, O) is an example of a *Nash equilibrium* (as is (F, F))

Mixed Equilibria – Rock-Paper-Scissors

	R	P	S
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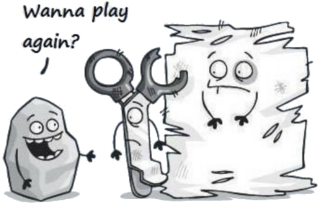

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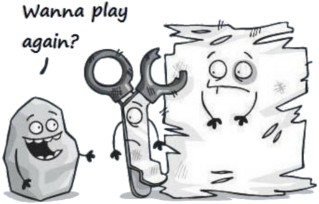

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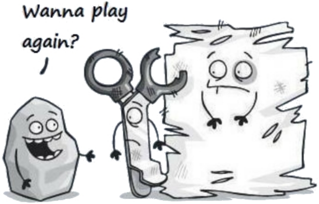

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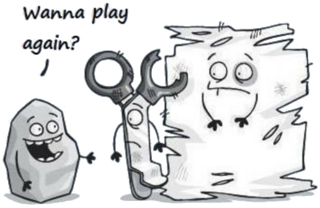

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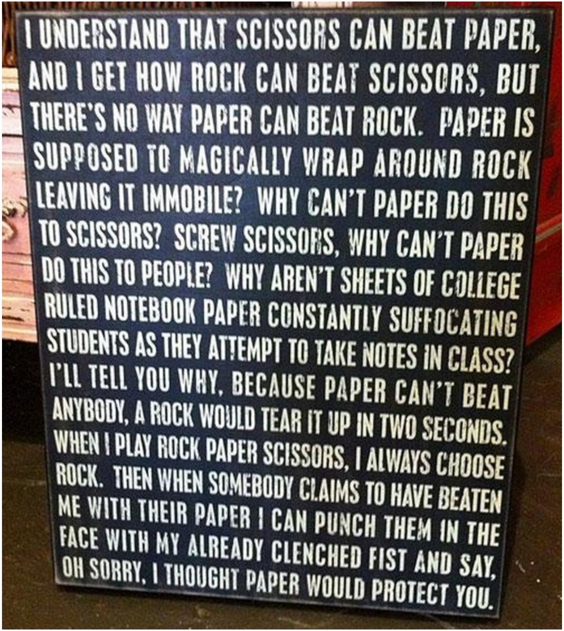
  Use *mixed strategies*: Each player plays each pure strategy with probability 1/3. The expected payoff of each player is 0 (even if one of the players changes his strategy, he still gets 0!).

Philosophical Issues in Games



Dynamic Games

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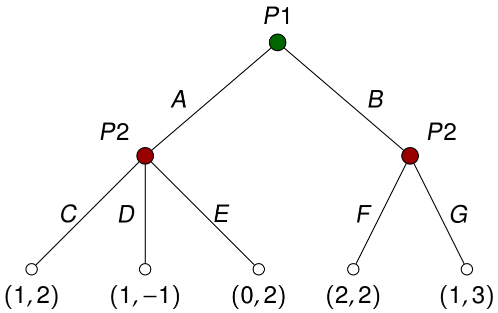

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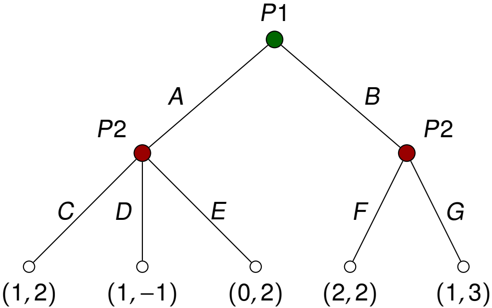

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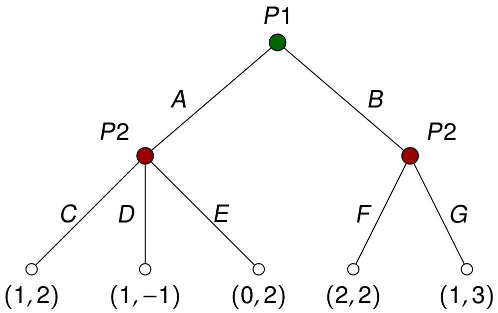

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What is their relationship to the strategic form games?

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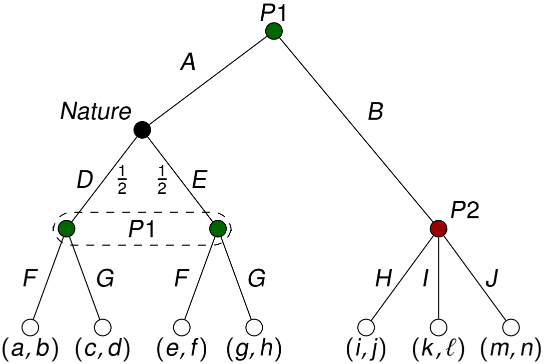

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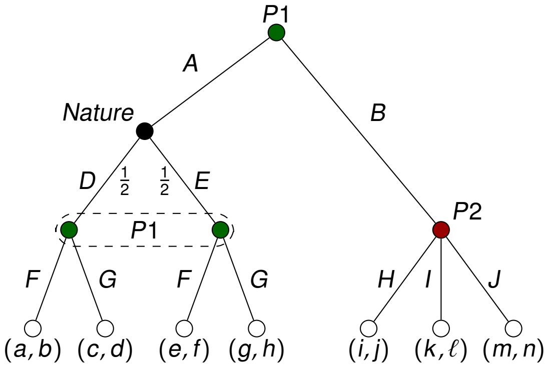

Again, how to solve such games?

Games of Incomplete Information

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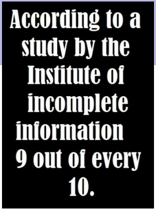

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$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

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How to deal with such a game? Assume the “worst” private value? What if we have a partial knowledge about the private values?

Inefficiency of Equilibria

In Prisoner's Dilemma, the selfish behavior of suspects (the Nash equilibrium) results in somewhat worse than ideal situation.

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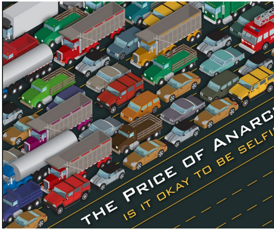
Price of Anarchy is the maximum ratio between values of equilibria and the value of an optimal solution.

Inefficiency of Equilibria – Selfish Routing

Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:



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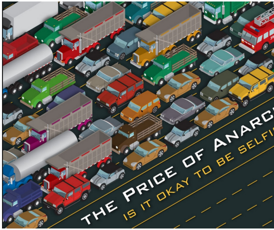

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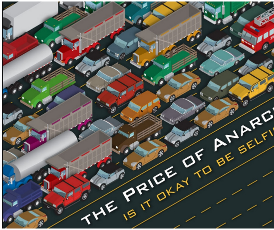

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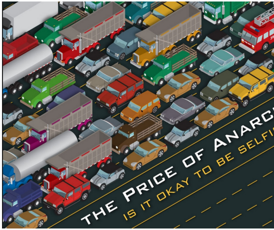

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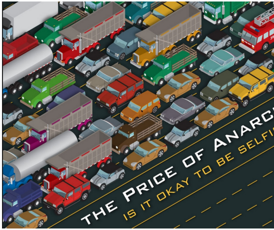

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Problem: Bound the price of anarchy over all routing games?

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- Games in machine learning: Generative adversarial networks, reinforcement learning

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- Games in Logic: modal and temporal logics, Ehrenfeucht-Fraisse games, etc.

Games in Computer Science

Games, the Internet and E-commerce: An extremely active research area at the intersection of CS and Economics

Basic idea: “The internet is a HUGE experiment in interaction between agents (both human and automated)”

How do we set up the rules of this game to harness “socially optimal” results?

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- Remaining time will be devoted to selected topics from extensive form games, games on graphs etc.

Static Games of Complete Information

Strategic-Form Games

Solution concepts

Static Games of Complete Information – Intuition

Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.

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Definition 1

A fact E is a *common knowledge* among players $\{1, \ldots, n\}$ if for every sequence $i_1, \ldots, i_k \in \{1, \ldots, n\}$ we have that i_1 knows that i_2 knows that ... i_{k-1} knows that i_k knows E.

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The goal of each player is to maximize his payoff (and this fact is a common knowledge).

Strategic-Form Games

To formally represent static games of complete information we define *strategic-form games*.

Definition 2

A game in *strategic-form* (or normal-form) is an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, in which:

- $N = \{1, 2, \ldots, n\}$ is a finite set of *players*.
- S_i is a set of *(pure) strategies* of player i, for every $i \in N$.

  A *strategy profile* is a vector of strategies of all players $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$.

  We denote the set of all strategy profiles by $S = S_1 \times \cdots \times S_n$.
- $u_i : S \to \mathbb{R}$ is a function associating each strategy profile $s = (s_1, \ldots, s_n) \in S$ with the *payoff* $u_i(s)$ to player i, for every player $i \in N$.

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Definition 3

A *zero-sum* game G is one in which for all $s = (s_1, \ldots, s_n) \in S$ we have $u_1(s) + u_2(s) + \cdots + u_n(s) = 0$.

Example: Prisoner's Dilemma

- $N = \{1, 2\}$
- $S_1 = S_2 = \{S, C\}$
- u_1, u_2 are defined as follows:
  - $u_1(C, C) = -5$, $u_1(C, S) = 0$, $u_1(S, C) = -20$, $u_1(S, S) = -1$
  - $u_2(C, C) = -5$, $u_2(C, S) = -20$, $u_2(S, C) = 0$, $u_2(S, S) = -1$

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(Is it zero sum?)

We usually write payoffs in the following form:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

or as two matrices:

	C	S
C	-5	0
S	-20	-1

	C	S
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S	0	-1

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Strategic-form game model $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1$
  $u_2(q_1, q_2) = q_2(\kappa - q_1 - q_2) - q_2 c_2$

Solution Concepts

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(I follow the approach of Steven Tadelis here, it is not completely standard)

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Example 4

Nash equilibrium is a solution concept. That is, we "solve" games by finding Nash equilibria and declare them to be reasonable outcomes.

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Here 4. implies non-cooperative game theory: Each player is in control of his actions, and he will stick to an action only if he finds it to be in his best interest.

Evaluating Solution Concepts

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- rationalizability
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For now, let us concentrate on

pure strategies only!

I.e., no mixed strategies are allowed. We will generalize to mixed setting later.

Notation

- Let $N = \{1, \ldots, n\}$ be a finite set and for each $i \in N$ let X_i be a set. Let $X := \prod_{i \in N} X_i = \{(x_1, \ldots, x_n) \mid x_j \in X_j, j \in N\}$.
  - For $i \in N$ we define $X_{-i} := \prod_{j \neq i} X_j$, i.e.,

    $$X_{-i} = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \mid x_j \in X_j, \forall j \neq i\}$$

  - An element of X_{-i} will be denoted by

    $$x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

    We slightly abuse notation and write (x_i, x_{-i}) to denote $(x_1, \ldots, x_i, \ldots, x_n) \in X$.

Strict Dominance in Pure Strategies

Definition 5
Let $s_i, s'_i \in S_i$ be strategies of player i. Then s'_i is *strictly dominated* by s_i (write $s_i \succ s'_i$) if for any possible profile of the other players' strategies, $s_{-i} \in S_{-i}$, we have

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

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Claim 1

An intelligent and rational player will never play a strictly dominated strategy.

Clearly, intelligence implies that the player should recognize dominated strategies, rationality implies that the player will avoid playing them.

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A strategy profile $s \in S$ is a *strictly dominant strategy equilibrium* if $s_i \in S_i$ is strictly dominant for all $i \in N$.

Corollary 8
If the strictly dominant strategy equilibrium exists, it is unique and rational players will play it.

Examples

In the Prisoner's dilemma:

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In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

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In the Battle of Sexes:

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no strictly dominant strategies exist.

Indiana Jones and the Last Crusade

(Taken from Dixit & Nalebuff's "The Art of Strategy" and a lecture of Robert Marks)

Indiana Jones, his father, and the Nazis have all converged at the site of the Holy Grail. The two Joneses refuse to help the Nazis reach the last step. So the Nazis shoot Indiana's dad. Only the healing power of the Holy Grail can save the senior Dr. Jones from his mortal wound. Suitably motivated, Indiana leads the way to the Holy Grail. But there is one final challenge. He must choose between literally scores of chalices, only one of which is the cup of Christ. While the right cup brings eternal life, the wrong choice is fatal. The Nazi leader impatiently chooses a beautiful gold chalice, drinks the holy water, and dies from the sudden death that follows from the wrong choice. Indiana picks a wooden chalice, the cup of a carpenter. Exclaiming "There's only one way to find out" he dips the chalice into the font and drinks what he hopes is the cup of life. Upon discovering that he has chosen wisely, Indiana brings the cup to his father and the water heals the mortal wound.

Indiana Jones and the Last Crusade (cont.)

Indy Goofed

- Although this scene adds excitement, it is somewhat embarrassing that such a distinguished professor as Dr. Indiana Jones would overlook his dominant strategy.
- He should have given the water to his father without testing it first.
  - If Indiana has chosen the right cup, his father is still saved.
  - If Indiana has chosen the wrong cup, then his father dies but Indiana is spared.
- Testing the cup before giving it to his father doesn't help, since if Indiana has made the wrong choice, there is no second chance – Indiana dies from the water and his father dies from the wound.

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Because it is common knowledge that all players will perform this kind of reasoning again, the process can continue until no more strictly dominated strategies can be eliminated.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

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1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are **not** strictly dominated in G_{DS}^k.

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Remark: If all S_i are *finite*, then in 2. we may remove only some of the strictly dominated strategies (not necessarily all). The result is *not* affected by the order of elimination since strictly dominated strategies remain strictly dominated even after removing some other strictly dominated strategies.

IESDS Examples

In the Prisoner's dilemma:

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all strategies survive all rounds (i.e. IESDS $\equiv$ anything may happen, sorry)

A Bit More Interesting Example

	L	C	R
L	$4, 3$	$5, 1$	$6, 2$
C	$2, 1$	$8, 4$	$3, 6$
R	$3, 0$	$9, 6$	$2, 8$

IESDS on greenboard!

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Hotelling (1929) and Downs (1957)

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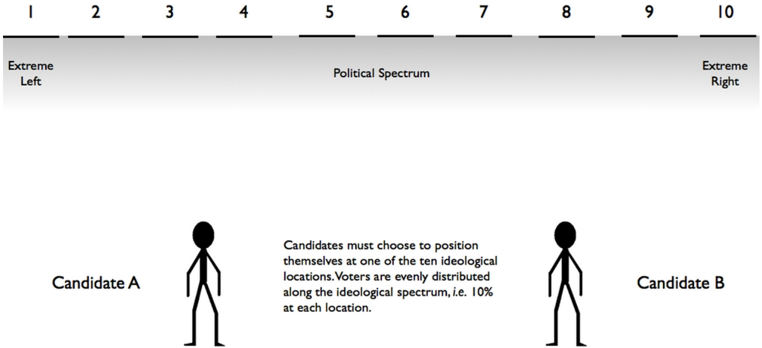
  (Here 10 means ten percent in the real-world)
- Voters vote for the closest candidate. If there is a tie, then $\frac{1}{2}$ got to each candidate
- Payoff: The number of voters for the candidate; each candidate (selfishly) strives to maximize this number

Political Science Example



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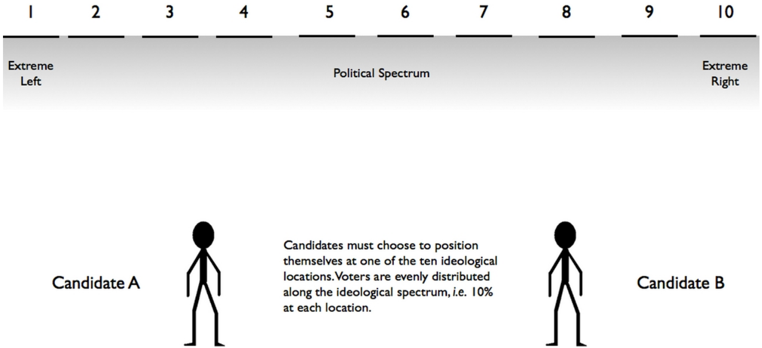

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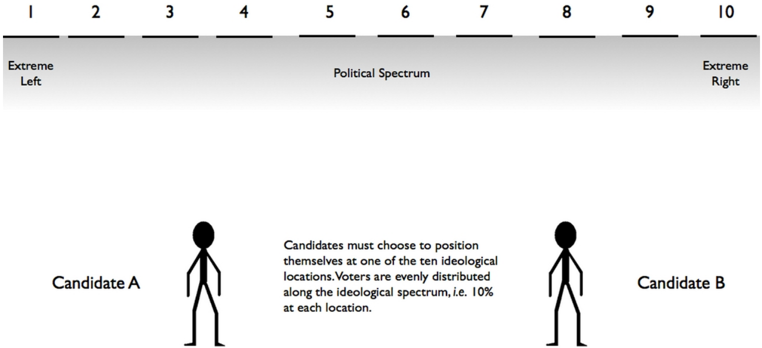

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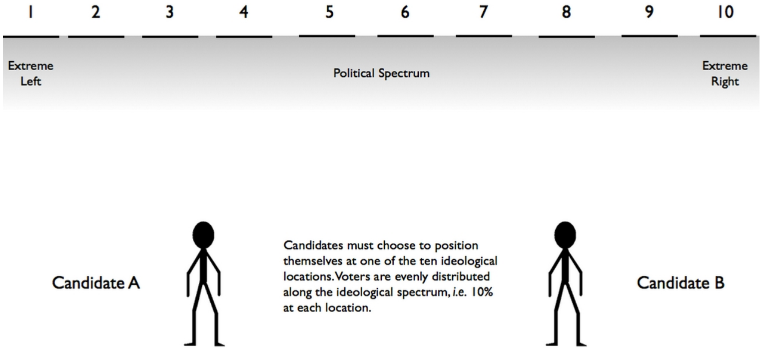

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- $\ldots$
- only $5, 6$ survive IESDS

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

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Let us formalize this type of reasoning...

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A strategy $s_i \in S_i$ of player i is a *best response* to a belief $s_{-i} \in S_{-i}$ if

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A rational player never plays any strategy that is never best response.

Best Response vs Strict Dominance

Proposition 1

If s_i is strictly dominated for player i, then it is never best response.

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The opposite does not have to be true in pure strategies:

	X	Y
A	$1, 1$	$1, 1$
B	$2, 1$	$0, 1$
C	$0, 1$	$2, 1$

Here A is never best response but is strictly dominated neither by B, nor by C.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
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(Warning: For some reasons, rationalizable strategies are almost always defined using mixed strategies!)

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all strategies are rationalizable.

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
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Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

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Thus $R_1^1 = R_2^1 = [0, \theta/2]$.

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Now, in G^1_{Rat}, we still have that $q_1 = (\theta - q_2)/2$ is the best response to q_2, and $q_2 = (\theta - q_1)/2$ the best resp. to q_1

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Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Now, in G^1_{Rat}, we still have that $q_1 = (\theta - q_2)/2$ is the best response to q_2, and $q_2 = (\theta - q_1)/2$ the best resp. to q_1

Since $q_2 \in R^1_2 = [0, \theta/2]$, we obtain that q_1 is never best response iff $q_1 \in [0, \theta/4)$
Similarly q_2 is never best response iff $q_2 \in [0, \theta/4)$

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

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Thus $R^2_1 = R^2_2 = [\theta/4, \theta/2]$.

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- $\ell_k = \theta/3 - \left(\frac{1}{4}\right)^k \theta/3$
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Hence, $\lim_{k\to\infty} \ell_k = \lim_{k\to\infty} r_k = \theta/3$ and thus $(\theta/3, \theta/3)$ is the only rationalizable equilibrium.

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Are $q_i = \theta/3$ the best outcomes possible? NO!

$$u_1(\theta/3, \theta/3) = u_2(\theta/3, \theta/3) = \theta^2/9$$

but

$$u_1(\theta/4, \theta/4) = u_2(\theta/4, \theta/4) = \theta^2/8$$

IESDS vs Rationalizability in Pure Strategies

Theorem 14
Assume that S *is finite. Then for all* k *we have that* $R_i^k \subseteq D_i^k$*. That is, in particular, all rationalizable strategies survive IESDS.*

IESDS vs Rationalizability in Pure Strategies

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Then $s'_i \in G^{k+1}_{Rat}$ since s'_i is *not* eliminated from G^k_{Rat}.
However, since s_i is a best response to s_{-i} in G^{k+1}_{Rat}, we get $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.
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By induction hypothesis, s_i is a best response to s_{-i} in G and the claim has been proved.

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Keep in mind: If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Now we prove $R_i^k \subseteq D_i^k$ for all players i by induction on k.

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Let $s_i \in R^{k+1}_i$. Then there must be $s_{-i} \in R^k_{-i}$ such that

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By the claim, s_i is a best response to s_{-i} in G as well!
By induction hypothesis, $s_i \in R^{k+1}_i \subseteq R^k_i \subseteq D^k_i$ and $s_{-i} \in R^k_{-i} \subseteq D^k_{-i}$.

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(This follows from the fact that the "best response" relationship of s_i and s_{-i} is preserved by removing arbitrarily many other strategies.)
Thus s_i is not strictly dominated in G^k_{DS} and $s_i \in D_i^{k+1}$. $\square$

Pinning Down Beliefs – Nash Equilibria

Criticism of previous approaches:

- Strictly dominant strategy equilibria often do not exist
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But are all strategy profiles really equally reasonable?

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(O,O) can be obtained as a profile where each player plays the best response to his belief and the **beliefs are correct**.

Nash Equilibrium

Nash equilibrium can be defined as a set of beliefs (one for each player) and a strategy profile in which every player plays a best response to his belief and each strategy of each player is consistent with beliefs of his opponents.

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A usual definition is following:

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A pure-strategy profile $s^* = (s_1^*, \dots, s_n^*) \in S$ is a (pure) Nash equilibrium if s_i^* is a best response to s_{-i}^* for each $i \in N$, that is

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Note that this definition is equivalent to the previous one in the sense that s_{-i}^* may be considered as the (consistent) belief of player i to which he plays a best response s_i^*

Nash Equilibria Examples

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In Cournot Duopoly, $(\theta/3, \theta/3)$ is the only Nash equilibrium.

(Best response relations: $q_1 = (\theta - q_2)/2$ and $q_2 = (\theta - q_1)/2$ are both satisfied only by $q_1 = q_2 = \theta/3$)

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Story:

- Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt

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If each player believes that the other one will go for hare, then (H, H) is a reasonable outcome $\Rightarrow$ a society of individualists who do not cooperate at all.

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Two NE: (S, S), and (H, H), where the former is strictly better for each player than the latter! Which one is more reasonable?

If each player believes that the other one will go for hare, then (H, H) is a reasonable outcome $\Rightarrow$ a society of individualists who do not cooperate at all.

If each player believes that the other will cooperate, then this anticipation is self-fulfilling and results in what can be called a cooperative society.

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

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This is supposed to explain that in real world there are societies that have similar endowments, access to technology and physical environment but have very different achievements, all because of self-fulfilling beliefs (or *norms* of behavior).

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(We will get to this *minimax* principle later)

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So it seems to be rational to expect (H, H) (?)

Nash Equilibria vs Previous Concepts

Theorem 16

1. *If s^* is a strictly dominant strategy equilibrium, then it is the unique Nash equilibrium.*
2. *Each Nash equilibrium is rationalizable and survives IESDS.*
3. *If S is finite, neither rationalizability, nor IESDS creates new Nash equilibria.*

Proof: Homework!

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Proof: Homework!

Corollary 17

Assume that S is finite. If rationalizability or IESDS result in a unique strategy profile, then this profile is a Nash equilibrium.

Interpretations of Nash Equilibria

Except the two definitions, usual interpretations are following:

- When the goal is to give advice to all of the players in a game (i.e., to advise each player what strategy to choose), any advice that was not an equilibrium would have the unsettling property that there would always be some player for whom the advice was bad, in the sense that, if all other players followed the parts of the advice directed to them, it would be better for some player to do differently than he was advised. If the advice is an equilibrium, however, this will not be the case, because the advice to each player is the best response to the advice given to the other players.

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- When the goal is prediction rather than prescription, a Nash equilibrium can also be interpreted as a potential stable point of a dynamic adjustment process in which individuals adjust their behavior to that of the other players in the game, searching for strategy choices that will give them better results.

Static Games of Complete Information
Mixed Strategies

Let's Mix It

As pointed out before, neither of the solution concepts has to exist in pure strategies

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Example: Rock-Paper-sCissors

	R	P	C
R	$0, 0$	$-1, 1$	$1, -1$
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How to solve this?

Let the players randomize their choice of pure strategies...

Before paper and scissors

Probability Distributions

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Let A be a finite set. A *probability distribution over A* is a function $\sigma : A \rightarrow [0,1]$ such that $\sum_{a \in A} \sigma(a) = 1$.

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Example 19

Consider $A = \{a, b, c\}$ and a function $\sigma : A \to [0,1]$ such that $\sigma(a) = \frac{1}{4}$, $\sigma(b) = \frac{3}{4}$, and $\sigma(c) = 0$. Then $\sigma \in \Delta(A)$.

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For example, in rock-paper-scissors, the pure strategy R corresponds to σ_i which satisfies $\sigma_i(X) = \begin{cases} 1 & X = R \\ 0 & \text{otherwise} \end{cases}$

Mixed Strategy Profiles

Let $\sigma = (\sigma_1, \sigma_2)$ be a mixed strategy profile.

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Thus for $s = (s_1, s_2) \in S = S_1 \times S_2$ we have that

$$\sigma(s) := \sigma_1(s_1) \cdot \sigma_2(s_2)$$

is the probability that the players randomly select the pure strategy profile s according to the mixed strategy profile σ.

(We abuse notation a bit here: σ denotes two things, a vector of mixed strategies as well as a probability distribution on S)

Mixed Strategies – Example

	R	P	C
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
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An example of a mixed strategy σ_1: $\sigma_1(R) = \frac{1}{2}$, $\sigma_1(P) = \frac{1}{3}$, $\sigma_1(C) = \frac{1}{6}$.

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Consider a mixed strategy profile (σ_1, σ_2) where $\sigma_1 = (\frac{1}{2}(R), \frac{1}{3}(P), \frac{1}{6}(C))$ and $\sigma_2 = (\frac{1}{3}(R), \frac{2}{3}(P), 0(C))$.

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Consider a mixed strategy profile (σ_1, σ_2) where
$\sigma_1 = (\frac{1}{2}(R), \frac{1}{3}(P), \frac{1}{6}(C))$ and $\sigma_2 = (\frac{1}{3}(R), \frac{2}{3}(P), 0(C))$.
Then the probability $\sigma(R, P)$ that the pure strategy profile (R, P) will be played by players playing the mixed profile (σ_1, σ_2) is

$$\sigma_1(R) \cdot \sigma_2(P) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Expected Payoff

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Definition 21
The *expected payoff* of player i under a mixed strategy profile $\sigma \in \Sigma$ is

$$u_i(\sigma) := \sum_{s \in S} \sigma(s) u_i(s) \qquad \left(= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdot u_i(s_1, s_2) \right)$$

I.e., it is the "weighted average" of what player i wins under each pure strategy profile s, weighted by the probability of that profile.

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I.e., it is the "weighted average" of what player i wins under each pure strategy profile s, weighted by the probability of that profile.

Assumption: Every rational player strives to maximize his own expected payoff.
(This assumption is not always completely convincing ...)

Expected Payoff – Example

Matching Pennies:

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Each player secretly turns a penny to heads or tails, and then they reveal their choices simultaneously. If the pennies match, player 1 (row) wins, if they do not match, player 2 (column) wins.

Consider $\sigma_1 = (\frac{1}{3}(H), \frac{2}{3}(T))$ and $\sigma_2 = (\frac{1}{4}(H), \frac{3}{4}(T))$

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &= \sum_{(X,Y)\in\{H,T\}^2} \sigma_1(X)\sigma_2(Y)u_1(X, Y) \\ &= \frac{1}{3}\frac{1}{4}1 + \frac{1}{3}\frac{3}{4}(-1) + \frac{2}{3}\frac{1}{4}(-1) + \frac{2}{3}\frac{3}{4}1 = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} u_2(\sigma_1, \sigma_2) &= \sum_{(X,Y)\in\{H,T\}^2} \sigma_1(X)\sigma_2(Y)u_2(X, Y) \\ &= \frac{1}{3}\frac{1}{4}(-1) + \frac{1}{3}\frac{3}{4}1 + \frac{2}{3}\frac{1}{4}1 + \frac{2}{3}\frac{3}{4}(-1) = -\frac{1}{6} \end{aligned}$$

Solution Concepts

We revisit the following solution concepts in mixed strategies:

- strict dominant strategy equilibrium
- IESDS equilibrium
- rationalizable equilibria
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We revisit the following solution concepts in mixed strategies:

- strict dominant strategy equilibrium
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- rationalizable equilibria
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mixed strategy.

In order to deal with efficiency issues we assume that the size of the game G is defined by $|G| := |N| + \sum_{i \in N} |S_i| + \sum_{i \in N} |u_i|$ where $|u_i| = \sum_{s \in S} |u_i(s)|$ and $|u_i(s)|$ is the length of a binary encoding of $u_i(s)$ (we assume that rational numbers are encoded as quotients of two binary integers)

Note that, in particular, $|G| > |S|$.

Strict Dominance in Mixed Strategies

Definition 22
Let $\sigma_1, \sigma_1' \in \Sigma_1$ be (mixed) strategies of player 1. Then σ_1' is *strictly dominated* by σ_1 (write $\sigma_1' \prec \sigma_1$) if

$$u_1(\sigma_1, s_2) > u_1(\sigma_1', s_2) \quad \text{for all } s_2 \in S_2$$

(Symmetrically for player 2.)

Comment: The above condition is equivalent to

$$u_1(\sigma_1, \sigma_2) > u_1(\sigma_1', \sigma_2) \quad \text{for all strategies } \sigma_2 \in \Sigma_2$$

Strict Dominance in Mixed Strategies

Example 23

	X	Y
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Is there a strictly dominated strategy?

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Question: Is there a game with at least one strictly dominated strategy but without strictly dominated *pure* strategies?

Strictly Dominant Strategy Equilibrium

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$\sigma_i \in \Sigma_i$ is *strictly dominant* if every other mixed strategy of player *i* is strictly dominated by σ_i.

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A strategy profile $\sigma \in \Sigma$ is a *strictly dominant strategy equilibrium* if $\sigma_i \in \Sigma_i$ is strictly dominant for each $i \in N$.

Proposition 2
If the strictly dominant strategy equilibrium exists, it is unique; all its strategies are pure, and rational players will play it.

Proof.
Homework. □

To compute the strictly dominant strategy equilibrium, it is sufficient to consider only pure strategies.

IESDS in Mixed Strategies

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting the pure strategy sets to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are *not* strictly dominated in G_{DS}^k by *mixed strategies*.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ *survives IESDS* if $s_i \in D_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 26
A strategy profile $s = (s_1, s_2) \in S$ is an *IESDS equilibrium* if both s_1 and s_2 survive IESDS.

Each D_i^{k+1} can be computed in polynomial time using *linear programming*.

IESDS in Mixed Strategie – Example

	X	Y
A	3	0
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Let us have a look at the first iteration of IESDS.

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Let us construct a set of constraints on mixed strategies (possibly) strictly dominating C:

$$
\begin{aligned}
3x_A + 0x_B + x_C &> 1 && \text{Row's payoff against } X \\
0x_A + 3x_B + x_C &> 1 && \text{Row's payoff against } Y \\
x_A, x_B, x_C &\geq 0 \\
x_A + x_B + x_C &= 1 && \text{x's must make a distribution}
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How to solve this?

Intermezzo: Linear Programming

Linear programming is a technique for optimization of a linear objective function, subject to linear (non-strict) inequality constraints.

Formally, a linear program in so called *canonical form* looks like this:

$$\text{maximize} \sum_{j=1}^{m} c_j x_j \qquad \textit{(objective function)}$$

$$\text{subject to} \sum_{j=1}^{m} a_{ij} x_j \le b_i \qquad 1 \le i \le n$$

$$x_j \ge 0 \qquad 1 \le j \le m \qquad \textit{(constraints)}$$

Here a_{ij}, b_k and c_j are real numbers and x_j's are real variables.

A *feasible solution* is an assignment of real numbers to the variables x_j, $1 \le j \le m$, so that the *constraints* are satisfied.

An *optimal solution* is a feasible solution which maximizes the *objective function* $\sum_{j=1}^{m} c_j x_j$.

Intermezzo: Complexity of Linear Programming

We assume that coefficients a_{ij}, b_k and c_j are encoded in binary (more precisely, as fractions of two integers encoded in binary).

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There is also a polynomial time algorithm (by Karmarkar) which has better complexity upper bounds than the Khachiyan's and sometimes works even better than the simplex.

Intermezzo: Complexity of Linear Programming

We assume that coefficients a_{ij}, b_k and c_j are encoded in binary (more precisely, as fractions of two integers encoded in binary).

Theorem 27 (Khachiyan, Doklady Akademii Nauk SSSR, 1979)

There is an algorithm which for any linear program computes an optimal solution in polynomial time.

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There exist several advanced linear programming solvers (usually parts of larger optimization packages) implementing various heuristics for solving large scale problems, sensitivity analysis, etc.

For more info see

http://en.wikipedia.org/wiki/Linear_programming#Solvers_and_scripting_.28programming.29_languages

IESDS in Mixed Strategie – Example

	X	Y
A	3	0
B	0	3
C	1	1

The linear program for deciding whether C is strictly dominated: The program maximizes y under the following constraints:

$$\begin{aligned} 3x_A + 0x_B + x_C &\geq 1 + y && \text{Row's payoff against } X \\ 0x_A + 3x_B + x_C &\geq 1 + y && \text{Row's payoff against } Y \\ x_A, x_B, x_C &\geq 0 && \\ x_A + x_B + x_C &= 1 && \text{x's must make a distribution} \\ y &\geq 0 && \end{aligned}$$

Here y just implements the strict inequality using $\geq$, we look for a solution with $y > 0$.

The maximum $y = \frac{1}{2}$ is attained at $x_A = \frac{1}{2}$ and $x_B = \frac{1}{2}$.

Note that in step 2 it is not sufficient to consider pure strategies. Consider the following zero sum game:

	X	Y
A	3	0
B	0	3
C	1	1

C is strictly dominated by $(\sigma_1(A), \sigma_1(B), \sigma_1(C)) = (\frac{1}{2}, \frac{1}{2}, 0)$ but no strategy is strictly dominated in pure strategies.

Best Response in Mixed Strategies

Definition 28
A *(mixed) belief* of player 1 is a mixed strategy σ_2 of player 2 (and vice versa).

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A *(mixed) belief* of player 1 is a mixed strategy σ_2 of player 2 (and vice versa).

Definition 29

$\sigma_1 \in \Sigma_1$ is a *best response* to a belief $\sigma_2 \in \Sigma_2$ if

$$u_1(\sigma_1, \sigma_2) \geq u_1(s_1, \sigma_2) \quad \text{for all } s_1 \in S_1$$

Denote by $BR_1(\sigma_2)$ the set of all best responses of player 1. (Symmetrically for player 2.)

Comment: The above condition is equivalent to

$$u_1(\sigma_1, \sigma_2) \geq u_1(\sigma_1', \sigma_2) \quad \text{for all } \sigma_1' \in \Sigma_1$$

Best Response – Example

Consider a game with the following payoffs of player 1:

	X	Y
A	2	0
B	0	2
C	1	1

- Player 1 (row) plays $\sigma_1 = (a(A), b(B), c(C))$.
- Player 2 (column) plays $(q(X), (1 - q)(Y))$ (we write just q).

Compute $BR_1(q)$.

Rationalizability in Mixed Strategies (Two Players)

Assumption: *A rational player* 1 *with a belief* σ_2 *always plays a best response to* σ_2 *(the same for player 2).*

Rationalizability in Mixed Strategies (Two Players)

Assumption: *A rational player* 1 *with a belief* σ_2 *always plays a best response to* σ_2 *(the same for player 2).*

Definition 30
A pure strategy $s_1 \in S_1$ of player 1 is *never best response* if it is not a best response to any belief σ_2 (similarly for player 2).

No rational player plays a strategy that is never best response.

Rationalizability in Mixed Strategies (Two Players)

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting the pure strategy sets to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are *best responses to some (mixed) beliefs in* G_{Rat}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ is *rationalizable* if $s_i \in R_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 31
A strategy profile $s = (s_1, s_2) \in S$ is a *rationalizable equilibrium* if both s_1 and s_2 are rationalizable.

Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
B	0	3
C	1	1

What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
B	0	3
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What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

Observation: The set of strictly dominated pure strategies coincides with the set of pure never best responses!

Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
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What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

Observation: The set of strictly dominated pure strategies coincides with the set of pure never best responses!

... and this holds in general for two player games:

Theorem 32

A pure strategy s_1 of player 1 is never best response to any belief σ_2 **iff** *s_1 is strictly dominated by a strategy $\sigma_1 \in \Sigma_1$ (similarly for player 2).*

It follows that a strategy of S_i survives IESDS **iff** it is rationalizable.

Mixed Nash Equilibrium

Definition 33

A mixed-strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ is a (mixed) Nash equilibrium if σ_1^* is a best response to σ_2^* and σ_2^* is a best response to σ_1^*. That is

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(s_1, \sigma_2^*) \quad \text{for all } s_1 \in S_1$$

$$u_2(\sigma_1^*, \sigma_2^*) \geq u_2(\sigma_1^*, s_2) \quad \text{for all } s_2 \in S_2$$

The above condition is equivalent to

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(\sigma_1, \sigma_2^*) \quad \text{for all } \sigma_1 \in \Sigma_1$$

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Theorem 34 (Nash 1950)

Every finite game in strategic form has a Nash equilibrium.

This is THE fundamental theorem of game theory.

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Player 1 (row) plays $(p(H), (1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1-q)(T))$ (we write q).

Compute all Nash equilibria.

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Compute all Nash equilibria.

What are the expected payoffs of playing pure strategies for player 1?

$$u_1(H, q) = 2q - 1 \text{ and } u_1(T, q) = 1 - 2q$$

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Then
$u_1(p, q) = pu_1(H, q) + (1-p)u_1(T, q) = p(2q-1) + (1-p)(1-2q)$.

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Then
$u_1(p, q) = pu_1(H, q) + (1-p)u_1(T, q) = p(2q-1) + (1-p)(1-2q).$

We obtain the best response correspondence BR_1:

$$BR_1(q) = \begin{cases} T & \text{if } q < \frac{1}{2} \\ p \in [0, 1] & \text{if } q = \frac{1}{2} \\ H & \text{if } q > \frac{1}{2} \end{cases}$$

Example: Matching Pennies

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

Player 1 (row) plays $(p(H),(1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H),(1-q)(T))$ (we write q).

Compute all Nash equilibria.

Similarly for player 2 :

$$u_2(p,H) = 1-2p \text{ and } u_2(p,T) = 2p-1$$

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Player 1 (row) plays $(p(H), (1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1-q)(T))$ (we write q).

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$$u_2(p, H) = 1 - 2p \text{ and } u_2(p, T) = 2p - 1$$

$$u_2(p, q) = qu_2(p, H) + (1-q)u_2(p, T) = q(1-2p) + (1-q)(2p-1)$$

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Compute all Nash equilibria.

Similarly for player 2 :

$$u_2(p,H) = 1-2p \text{ and } u_2(p,T) = 2p-1$$

$u_2(p,q) = qu_2(p,H) + (1-q)u_2(p,T) = q(1-2p) + (1-q)(2p-1)$
We obtain best-response relation BR_2:

$$BR_2(p) = \begin{cases} H & \text{if } p < \frac{1}{2} \\ q \in [0,1] & \text{if } p = \frac{1}{2} \\ T & \text{if } p > \frac{1}{2} \end{cases}$$

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Player 1 (row) plays $(p(H), (1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1-q)(T))$ (we write q).

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We obtain best-response relation BR_2:

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The only "intersection" of BR_1 and BR_2 is the only Nash equilibrium $\sigma_1 = \sigma_2 = (\frac{1}{2}, \frac{1}{2})$.

Support Enumeration

Computing Mixed Nash Equilibria

Lemma 35

Every Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ *satisfies*

- $u_1(s_1, \sigma_2^*) = u_1(\sigma^*)$ *for* $s_1 \in supp(\sigma_1^*)$
- $u_2(\sigma_1^*, s_2) = u_2(\sigma^*)$ *for* $s_2 \in supp(\sigma_2^*)$

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Proof. W.l.o.g. consider only the player 1 and assume that σ^* is a Nash equilibrium.

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The latter assumption implies $u_1(s_1, \sigma_2^*) \leq u_1(\sigma^*)$ for all $s_1 \in S_1$.

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The latter assumption implies $u_1(s_1, \sigma_2^*) \leq u_1(\sigma^*)$ for all $s_1 \in S_1$.

Now, if there exists $s_1' \in supp(\sigma_1^*) \subseteq S_1$ satisfying $u_1(s_1', \sigma_2^*) < u_1(\sigma^*)$, then because $\sigma_1^*(s_1') > 0$ we have

$$u_1(\sigma^*) = \sum_{s_1 \in S_1} \sigma_1^*(s_1) u_1(s_1, \sigma_2^*) < \sum_{s_1 \in S_1} \sigma_1^*(s_1) u_1(\sigma^*) = u_1(\sigma^*)$$

A contradiction.

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A contradiction.

Thus $u_1(s_1, \sigma_2^*) = u_1(\sigma^*)$ for all $s_1 \in supp(\sigma_1^*)$.

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Player 1 (row) plays $(p(H), (1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1-q)(T))$ (we write q).

Compute all Nash equilibria.

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There are no equilibria where only player 1 randomizes:

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There are no equilibria where only player 1 randomizes:

Indeed, assume that (p, H) is such an equilibrium. Then by Lemma 35,

$$1 = u_1(H, H) = u_1(T, H) = -1$$

a contradiction. Also, (p, T) cannot be an equilibrium.

Similarly, there is no NE where only player 2 randomizes.

Example: Matching Pennies

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T	$-1, 1$	$1, -1$

Player 1 (row) plays $(p(H), (1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1-q)(T))$ (we write q).

Compute all Nash equilibria.

Assume that both players randomize, i.e., $p, q \in (0, 1)$.

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Compute all Nash equilibria.

Assume that both players randomize, i.e., $p, q \in (0, 1)$.

The expected payoffs of playing pure strategies for player 1:

$$u_1(H, q) = 2q - 1 \text{ and } u_1(T, q) = 1 - 2q$$

Similarly for player 2 :

$$u_2(p, H) = 1 - 2p \text{ and } u_1(p, T) = 2p - 1$$

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$$u_2(p, H) = 1 - 2p \text{ and } u_1(p, T) = 2p - 1$$

By Lemma 35, such Nash equilibria must satisfy:

$$2q - 1 = 1 - 2q \qquad \text{and} \qquad 1 - 2p = 2p - 1$$

That is $p = q = \frac{1}{2}$ is the only Nash equilibrium.

Example: Battle of Sexes

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Player 1 (row) plays $(p(O), (1-p)(F))$ (we write just p) and player 2 (column) plays $(q(O), (1-q)(F))$ (we write q).

Compute all Nash equilibria.

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	O	F
O	$2, 1$	$0, 0$
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Player 1 (row) plays $(p(O), (1 - p)(F))$ (we write just p) and player 2 (column) plays $(q(O), (1 - q)(F))$ (we write q).

Compute all Nash equilibria.

There are two pure strategy equilibria (O, O) and (F, F), no Nash equilibrium where only one player randomizes.

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Compute all Nash equilibria.

There are two pure strategy equilibria (O,O) and (F,F), no Nash equilibrium where only one player randomizes.

Now assume that

- player 1 (row) plays $(p(O),(1-p)(F))$ (we write just p) and
- player 2 (column) plays $(q(O),(1-q)(F))$ (we write q)

where $p,q \in (0,1)$.

Example: Battle of Sexes

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O	$2,1$	$0,0$
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Player 1 (row) plays $(p(O),(1-p)(F))$ (we write just p) and player 2 (column) plays $(q(O),(1-q)(F))$ (we write q).

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By Lemma 35, such Nash equilibria must satisfy:

$$2q = 1-q \qquad \text{and} \qquad p = 2(1-p)$$

This holds only for $q = \frac{1}{3}$ and $p = \frac{2}{3}$.

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What did we do in the previous examples?

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Whenever one of the *supports* was non-singleton, we reduced computation of Nash equilibria to *linear equations*.

Computing Mixed Nash Equilibria

Lemma 36

Let $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ *be a mixed profile. Assume that there exist* $w_1, w_2 \in \mathbb{R}$ *such that*

- $u_1(s_1, \sigma_2^*) = w_1$ *for* $s_1 \in supp(\sigma_1^*)$
- $u_1(s_1, \sigma_2^*) \leq w_1$ *for* $s_1 \notin supp(\sigma_1^*)$
- $u_2(\sigma_1^*, s_2) = w_2$ *for* $s_2 \in supp(\sigma_2^*)$
- $u_2(\sigma_1^*, s_2) \leq w_2$ *for* $s_2 \notin supp(\sigma_2^*)$

Then $u_1(\sigma^*) = w_1$ *and* $u_2(\sigma^*) = w_2$, *and* σ^* *is a Nash equilibrium.*

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- $u_1(s_1, \sigma_2^*) \leq w_1$ *for* $s_1 \notin supp(\sigma_1^*)$
- $u_2(\sigma_1^*, s_2) = w_2$ *for* $s_2 \in supp(\sigma_2^*)$
- $u_2(\sigma_1^*, s_2) \leq w_2$ *for* $s_2 \notin supp(\sigma_2^*)$

Then $u_1(\sigma^*) = w_1$ and $u_2(\sigma^*) = w_2$, and σ^* is a Nash equilibrium.

Proof. Consider just the player 1 (for pl. 2 similarly):

$$
\begin{aligned}
u_1(\sigma^*) &= \sum_{s_1 \in S_1} \sigma^*(s_1) u_1(s_1, \sigma_2^*) = \sum_{s_1 \in supp(\sigma_1^*)} \sigma^*(s_1) u_1(s_1, \sigma_2^*) \\
&= \sum_{s_1 \in supp(\sigma_1^*)} \sigma^*(s_1) w_1 = w_1 \sum_{s_1 \in supp(\sigma_1^*)} \sigma^*(s_1) = w_1
\end{aligned}
$$

Now the fact that σ^* is a Nash equilibrium follows from the definition.

How to Compute Mixed Nash Equilibria?

Every Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*)$ can be computed by finding appropriate w_1, w_2 so that

- $u_1(s_1, \sigma_2^*) = w_1$ for $s_1 \in supp(\sigma_1^*)$
- $u_1(s_1, \sigma_2^*) \leq w_1$ for $s_1 \notin supp(\sigma_1^*)$
- $u_2(\sigma_1^*, s_2) = w_2$ for $s_2 \in supp(\sigma_2^*)$
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Indeed,

- by Lemma 36, all σ^* and w_1, w_2 satisfying the above inequalities give a Nash equilibrium σ^* with $u_1(\sigma^*) = w_1$ and $u_2(\sigma^*) = w_2$,
- by Lemma 35, for every Nash equilibrium σ^* choosing $w_1 = u_1(\sigma^*)$ and $w_2 = u_2(\sigma^*)$ satisfies the above inequalities.

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- by Lemma 35, for every Nash equilibrium σ^* choosing $w_1 = u_1(\sigma^*)$ and $w_2 = u_2(\sigma^*)$ satisfies the above inequalities.

Suppose that we somehow know the supports $supp(\sigma_1^*), supp(\sigma_2^*)$ for some Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*)$ (which itself is unknown to us).

We may consider all $\sigma_i^*(s_i)$'s and both w_1, w_2's as variables and use the above conditions to design a system of inequalities capturing Nash equilibria with the given support sets $supp(\sigma_1^*), supp(\sigma_2^*)$.

Support Enumeration

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Then $\sigma_i(j)$ is the probability of the pure strategy j in the mixed strategy σ_i.

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1. For all $k \in supp_1$ and all $\ell \in supp_2$:

$$\sum_{\ell' \in S_2} \sigma_2(\ell') u_1(k, \ell') = w_1 \qquad \sum_{k' \in S_1} \sigma_1(k') u_2(k', \ell) = w_2$$

2. For all $k \notin supp_1$ and all $\ell \notin supp_2$:

$$\sum_{\ell' \in S_2} \sigma_2(\ell') u_1(k, \ell') \leq w_1 \qquad \sum_{k' \in S_1} \sigma_1(k') u_2(k', \ell) \leq w_2$$

3. For all $i \in \{1, 2\}$: $\sigma_i(1) + \cdots + \sigma_i(m_i) = 1$.
4. For all $i \in \{1, 2\}$ and all $k \in supp_i$: $\sigma_i(k) \geq 0$.
5. For all $i \in \{1, 2\}$ and all $k \notin supp_i$: $\sigma_i(k) = 0$.

Support Enumeration

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Input: A two-player strategic-form game G with strategy sets $S_1 = \{1, \dots, m_1\}$ and $S_2 = \{1, \dots, m_2\}$ and rational payoffs u_1, u_2.

Output: A Nash equilibrium σ^*.

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Output: A Nash equilibrium σ^*.

Algorithm: For all possible $supp_1 \subseteq S_1$ and $supp_2 \subseteq S_2$:

- Check if the corresponding system of linear constraints (from the previous slide) has a feasible solution σ^*, w_1^*, w_2^*.
- If so, STOP: the feasible solution σ^* is a Nash equilibrium satisfying $u_i(\sigma^*) = w_i^*$.

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Question: How many possible subsets $supp_1, supp_2$ are there to try?

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- If so, STOP: the feasible solution σ^* is a Nash equilibrium satisfying $u_i(\sigma^*) = w_i^*$.

Question: How many possible subsets $supp_1, supp_2$ are there to try?
Answer: $2^{(m_1+m_2)}$

So, unfortunately, the algorithm requires worst-case exponential time.

Remarks on Support Enumeration

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  (There are algorithms for computing (a finite representation of) a set of all feasible solutions of a given linear constraint system.)
- The algorithm can be used to compute "good" equilibria.

  For example, to find a Nash equilibrium maximizing the sum of all expected payoffs (the "social welfare") it suffices to solve the system of constraints while maximizing $w_1 + w_2$. More precisely, the algorithm can be modified as follows:
  - Initialize $W := -\infty$ (W stores the current maximum welfare)
  - For all possible $supp_1 \subseteq S_1$ and $supp_2 \subseteq S_2$:
    - Find the maximum value $\max(w_1 + w_2)$ of $w_1 + w_2$ so that the constraints are satisfiable (using linear programming).
    - Put $W := \max\{W, \max(w_1 + w_2)\}$.
  - Return W.

Remarks on Support Enumeration (Cont.)

Similar trick works for any notion of "good" NE that can be expressed using a linear objective function and (additional) linear constraints in variables $\sigma_i(j)$ and w_i.

(e.g., maximize payoff of player 1, minimize payoff of player 2 and keep probability of playing the strategy 1 below $1/2$, etc.)

Complexity Results – (Two Players)

Theorem 37
Given a two-player game in strategic form, a mixed Nash equilibrium can be computed in exponential time.

Theorem 38
All the following problems are NP-complete: Given a two-player game in strategic form, does it have

1. *a NE in which player 1 has utility at least a given amount v ?*
2. *a NE in which the sum of expected payoffs of the two players is at least a given amount v ?*
3. *a NE with a support of size greater than a given number?*
4. *a NE whose support contains a given strategy s ?*
5. *a NE whose support does not contain a given strategy s ?*
6. *....*

NP-hardness can be proved using reduction from SAT.

The Reduction (It's Short and Sweet)

Definition 4 *Let ϕ be a Boolean formula in conjunctive normal form (representing a SAT instance). Let V be its set of variables (with $|V| = n$), L the set of corresponding literals (a positive and a negative one for each variable*[6]*), and C its set of clauses. The function $v : L \to V$ gives the variable corresponding to a literal,* e.g., *$v(x_1) = v(-x_1) = x_1$. We define $G_\epsilon(\phi)$ to be the following finite symmetric 2-player game in normal form. Let $\Sigma = \Sigma_1 = \Sigma_2 = L \cup V \cup C \cup \{f\}$. Let the utility functions be*

- $u_1(l^1, l^2) = u_2(l^2, l^1) = n - 1$ *for all* $l^1, l^2 \in L$ *with* $l^1 \neq -l^2$;
- $u_1(l, -l) = u_2(-l, l) = n - 4$ *for all* $l \in L$;
- $u_1(l, x) = u_2(x, l) = n - 4$ *for all* $l \in L$, $x \in \Sigma - L - \{f\}$;
- $u_1(v, l) = u_2(l, v) = n$ *for all* $v \in V$, $l \in L$ *with* $v(l) \neq v$;
- $u_1(v, l) = u_2(l, v) = 0$ *for all* $v \in V$, $l \in L$ *with* $v(l) = v$;
- $u_1(v, x) = u_2(x, v) = n - 4$ *for all* $v \in V$, $x \in \Sigma - L - \{f\}$;
- $u_1(c, l) = u_2(l, c) = n$ *for all* $c \in C$, $l \in L$ *with* $l \notin c$;
- $u_1(c, l) = u_2(l, c) = 0$ *for all* $c \in C$, $l \in L$ *with* $l \in c$;
- $u_1(c, x) = u_2(x, c) = n - 4$ *for all* $c \in C$, $x \in \Sigma - L - \{f\}$;
- $u_1(x, f) = u_2(f, x) = 0$ *for all* $x \in \Sigma - \{f\}$;
- $u_1(f, f) = u_2(f, f) = \epsilon$;
- $u_1(f, x) = u_2(x, f) = n - 1$ *for all* $x \in \Sigma - \{f\}$.

Theorem 1 *If $(l_1, l_2, \ldots, l_n)$ (where $v(l_i) = x_i$) satisfies ϕ, then there is a Nash equilibrium of $G_\epsilon(\phi)$ where both players play l_i with probability $\frac{1}{n}$, with expected utility $n - 1$ for each player. The only other Nash equilibrium is the one where both players play f, and receive expected utility ϵ each.*

... But What is The Exact Complexity of *Computing* Nash Equilibria in Two Player Games?

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As the class NP consists of decision problems, it cannot be directly used to characterize complexity of the sample equilibrium problem.

We use complexity classes of *function problems* such as FP, FNP, etc. The sample equilibrium problem belongs to the complexity class PPAD (which is a subclass of TFNP) for two-player games.

A binary relation P(x,y) is in TFNP if and only if there is a deterministic polynomial time algorithm that can determine whether P(x,y) holds given both x and y, and for every x, there exists a y which is at most polynomially longer than x such that P(x,y) holds.

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Can we do better than FNP (i.e. exponential time)?

In what follows we show that the sample equilibrium problem can be solved in polynomial time for zero-sum two-player games.
(Using a beautiful characterization of all Nash equilibria)

MaxMin

Definition 39

$\sigma_1^* \in \Sigma_1$ is a *maxmin* strategy of player 1 if

$$\sigma_1^* \in \operatorname*{argmax}_{\sigma_1 \in \Sigma_1} \min_{s_2 \in S_2} u_1(\sigma_1, s_2) \quad \left(= \operatorname*{argmax}_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2)\right)$$

(Intuitively, a *maxmin* strategy σ_1^* maximizes player 1's worst-case payoff in the situation where player 2 strives to cause the greatest harm to player 1.)

Similarly, $\sigma_2^* \in \Sigma_2$ is a *maxmin* strategy of player 2 if

$$\sigma_2^* \in \operatorname*{argmax}_{\sigma_2 \in \Sigma_2} \min_{s_1 \in S_1} u_2(s_1, \sigma_2)$$

Which, assuming zero-sum games, i.e. $u_1 = -u_2$, becomes

$$\sigma_2^* \in \operatorname*{argmin}_{\sigma_2 \in \Sigma_2} \max_{s_1 \in S_1} u_1(s_1, \sigma_2) \quad \left(= \operatorname*{argmin}_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2)\right)$$

Note that the payoff function is the same for both players!!

Zero-Sum Games: von Neumann's Theorem

Theorem 40 (von Neumann)

Assume a two-player **zero-sum** *game. Then*

$$\max_{\sigma_1 \in \Sigma_1} \min_{s_2 \in S_2} u_1(\sigma_1, s_2) = \min_{\sigma_2 \in \Sigma_2} \max_{s \in S_1} u_1(s_1, \sigma_2)$$

Morever, $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ *is a Nash equilibrium* **iff** *both* σ_1^* *and* σ_2^* *are maxmin.*

So to compute a Nash equilibrium it suffices to compute (arbitrary) maxmin strategies for both players.

Zero-Sum Two-Player Games – Computing NE

Assume $S_1 = \{1, \ldots, m_1\}$ and $S_2 = \{1, \ldots, m_2\}$.

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Consider a linear program with variables $\sigma_1(1), \ldots, \sigma_1(m_1), v$:

$$\begin{array}{rll}
\textbf{maximize:} & v & \\
\textbf{subject to:} & \displaystyle\sum_{k=1}^{m_1} \sigma_1(k) \cdot u_1(k, \ell) \geq v & \ell = 1, \ldots, m_2 \\
& \displaystyle\sum_{k=1}^{m_1} \sigma_1(k) = 1 & \\
& \sigma_1(k) \geq 0 & k = 1, \ldots, m_1
\end{array}$$

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& && \sum_{k=1}^{m_1} \sigma_1(k) = 1 && \\
& && \sigma_1(k) \geq 0 && k = 1, \dots, m_1
\end{aligned}$$

Lemma 41

$\sigma_1^* \in \operatorname{argmax}_{\sigma_1 \in \Sigma_1} \min_{\ell \in S_2} u_1(\sigma_1, \ell)$ **iff** *assigning* $\sigma_1(k) := \sigma_1^*(k)$ *and* $v := \min_{\ell \in S_2} u_1(\sigma_1^*, \ell)$ *gives an optimal solution.*

Zero-Sum Two-Player Games – Computing NE

Summary:

- We have reduced computation of NE to computation of maxmin strategies for both players.
- Maxmin strategies can be computed using linear programming in polynomial time.
- That is, Nash equilibria in zero-sum two-player games can be computed in polynomial time.

Strategic-Form Games – Conclusion

We have considered *static games of complete information*, i.e., "one-shot" games where the players know exactly what game they are playing.
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We modeled such games using *strategic-form games*.

We have considered both pure strategy setting and mixed strategy setting.

In both cases, we considered four solution concepts:

- Strictly dominant strategies
- Iterative elimination of strictly dominated strategies
- Rationalizability (i.e., iterative elimination of strategies that are never best responses)
- Nash equilibria

Strategic-Form Games – Conclusion

In pure strategy setting:

1. Strictly dominant strategy equilibrium survives IESDS, rationalizability and is the unique Nash equilibrium (if it exists)
2. In finite games, rationalizable equilibria survive IESDS, IESDS preserves the set of Nash equilibria
3. In finite games, rationalizability preserves Nash equilibria

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In mixed setting:

1. In finite two player games, IESDS and rationalizability coincide.
2. Strictly dominant strategy equilibrium survives IESDS (rationalizability) and is the unique Nash equilibrium (if it exists)
3. In finite games, IESDS (rationalizability) preserves Nash equilibria

The proofs for 2. and 3. in the mixed setting are similar to corresponding proofs in the pure setting.

Algorithms

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- Strictly dominant strategy equilibria coincide in pure and mixed settings, and can be computed in polynomial time.
- IESDS and rationalizability can be implemented in polynomial time in the pure setting as well as in the mixed setting

  In the mixed setting, linear programming is needed to implement one step of IESDS (rationalizability).
- Nash equilibria can be computed for two-player games
  - in polynomial time for zero-sum games
    (using von Neumann's theorem and linear programming)
  - in exponential time using support enumeration
  - in PPAD using Lemke-Howson (omitted)

Loose Ends – Modes of Dominance

To simplify, let us consider only **pure strategies.**

Let $s_i, s_i' \in S_i$. Then s_i' is *strictly dominated* by s_i if
$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad$ for all $s_{-i} \in S_{-i}$.

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Let $s_i, s_i' \in S_i$. Then s_i' is *weakly dominated* by s_i if
$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$ and there is $s'_{-i} \in S_{-i}$ such that $u_i(s_i, s'_{-i}) > u_i(s_i', s'_{-i})$.

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Loose Ends – Modes of Dominance

To simplify, let us consider only **pure strategies**.

Let $s_i, s'_i \in S_i$. Then s'_i is *strictly dominated* by s_i if
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Claim 4

Any pure strategy profile $s \in S$ such that each s_i is very weakly dominant is a Nash equilibrium.

The same claim can be proved in the mixed strategy setting.

Dynamic Games of Complete Information

Extensive-Form Games

Definition

Sub-Game Perfect Equilibria

Dynamic Games of Perfect Information (Motivation)

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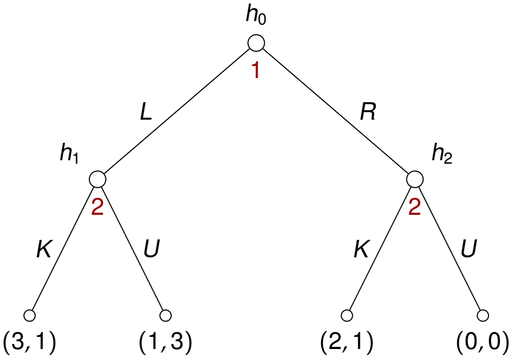
Then generalize to imperfect information, where players may have only partial knowledge of these results (e.g., most card games).

Perfect-Info. Extensive-Form Games (Example)



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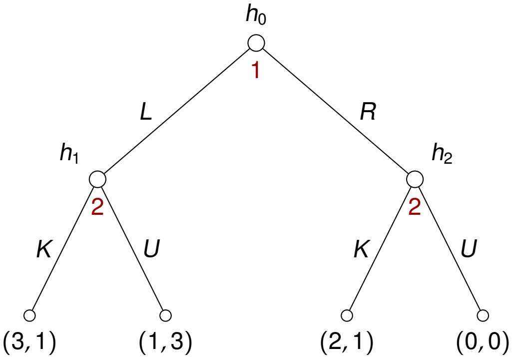

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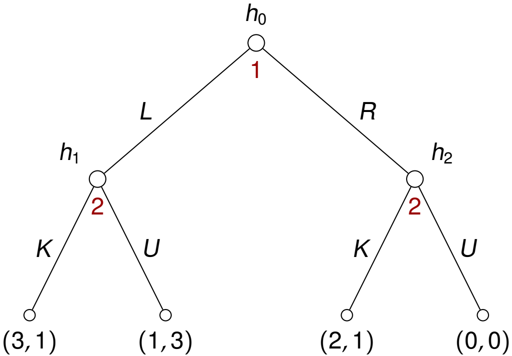

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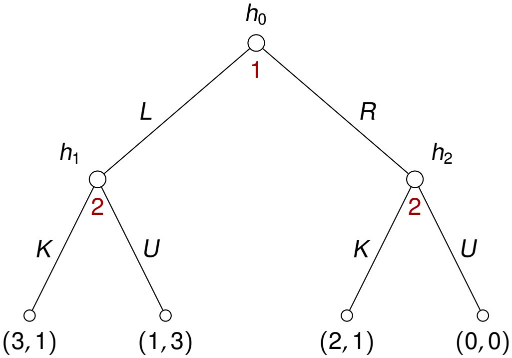

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Every action results in a transition to a new node.
Choosing L in h_0 results in a move to h_1
When a play reaches a terminal node, players collect payoffs.
E.g., the left most terminal node gives 3 to player 1 and 1 to player 2.

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- $u = (u_1, \ldots, u_n)$, where each $u_i : Z \to \mathbb{R}$ is a *payoff function* for player i in the terminal nodes of Z.

Extensive-Form Games as Rooted Trees

h' is a *child* of h, and h is a *parent* of h' if there is $a \in \chi(h)$ such that $h' = \pi(h, a)$.

A *path* from $h \in \mathcal{H}$ to $h' \in \mathcal{H}$ is a sequence $h_1 a_2 h_2 a_3 h_3 \cdots h_{k-1} a_k h_k$ where $h_1 = h$, $h_k = h'$ and $\pi(h_{j-1}, a_j) = h_j$ for every $1 < j \leq k$.
Note that, in particular, h is a path from h to h.

$h' \in \mathcal{H}$ is *reachable* from $h \in \mathcal{H}$ if there is a path from h to h'.
If h' is reachable from h we say that h' is a descendant of h and h is an ancestor of h'

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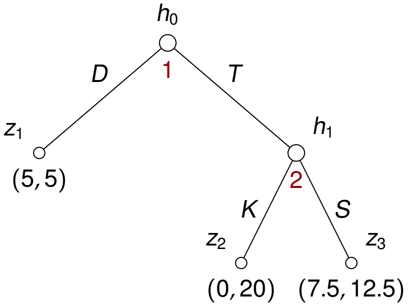
Every perfect-information extensive-form game can be seen as a game on a *rooted tree* $(\mathcal{H}, E, h_0)$ where

- $H \cup Z$ is a set of nodes,
- $E \subseteq \mathcal{H} \times \mathcal{H}$ is a set of edges defined by $(h, h') \in E$ iff $h \in H$ and there is $a \in \chi(h)$ such that $\pi(h, a) = h'$,
- h_0 is the root.

Example: Trust Game



- Two players, both start with 5$

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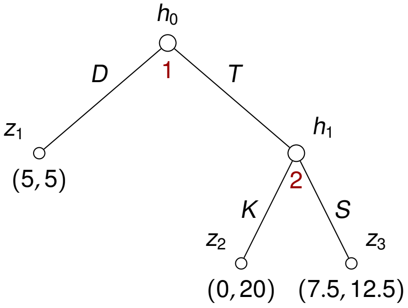

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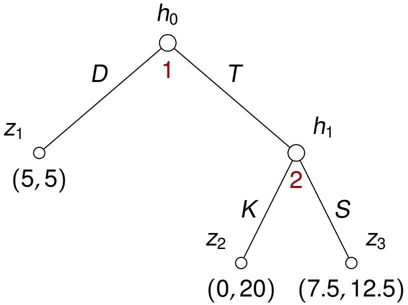

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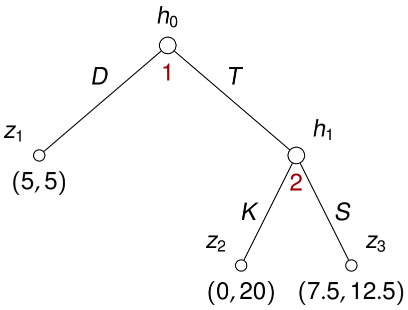

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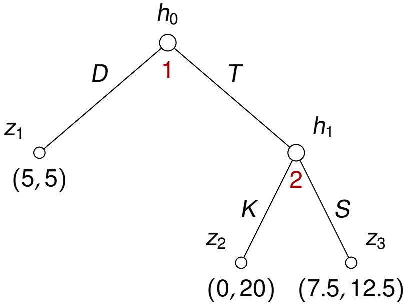
- Player 2 may either keep (K) the additional 15$ (resulting in $(0, 20)$), or share (S) it with player 1 (resulting in $(7.5, 12.5)$)

Example: Trust Game (Cont.)



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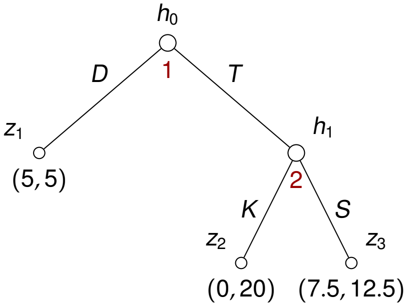

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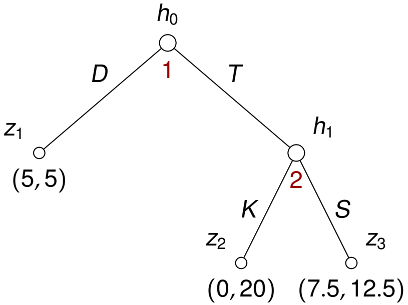

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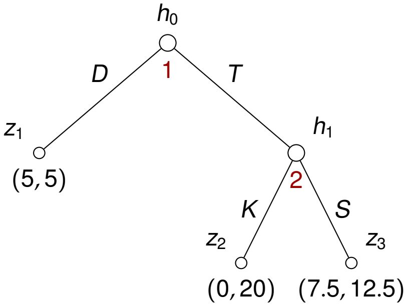

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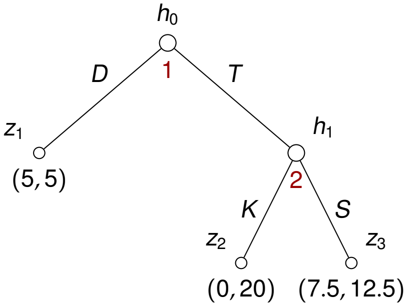

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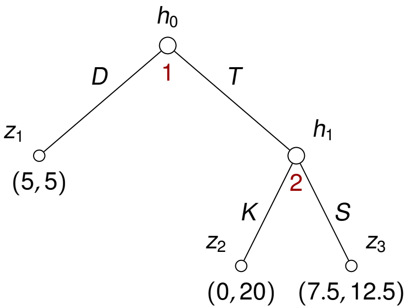

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- $u_1(z_1) = 5$, $u_1(z_2) = 0$, $u_1(z_3) = 7.5$, $u_2(z_1) = 5$, $u_2(z_2) = 20$, $u_2(z_3) = 12.5$

Stackelberg Competition

Very similar to Cournot duopoly ...

- Two identical firms, players 1 and 2, produce some good.
  Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.
- The price of each item is $\kappa - q_1 - q_2$ where $\kappa > 0$ is fixed.
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Except that ...

- As opposed to Cournot duopoly, the firm 1 moves first, and chooses the quantity $q_1 \in [0, \infty)$.
- Afterwards, the firm 2 chooses $q_2 \in [0, \infty)$ (knowing q_1) and then the firms get their payoffs.

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- $h_0 = (b_0, 1)$ where b_0 is the initial board
- $u_j(wb, i) \in \{1, 0, -1\}$, here 1 means "win", 0 means "draw", and -1 means "loss" for player j

Pure Strategies

Let $G = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$ be a perfect-information extensive-form game.

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Note that each pure strategy profile $s \in S$ determines a unique path $w_s = h_0 a_1 h_1 \cdots h_{k-1} a_k h_k$ from h_0 to a terminal node h_k by

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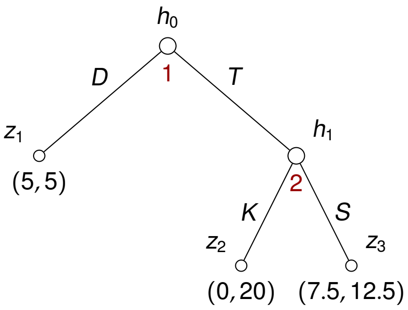
Denote by $O(s)$ the terminal node reached by w_s.

Abusing notation a bit, we denote by $u_i(s)$ the value $u_i(O(s))$ of the payoff for player i when the terminal node $O(s)$ is reached using strategies of s.

Example: Trust Game



A pure strategy profile (s_1, s_2) where

$$s_1(h_0) = T \qquad \text{and} \qquad s_2(h_1) = K$$

is usually written as TK (BFS & left to right traversal) determines the path $h_0 T\, h_1 K\, z_2$

The resulting payoffs: $u_1(s_1, s_2) = 0$ and $u_2(s_1, s_2) = 20$.

Extensive-Form vs Strategic-Form

The extensive-form game G determines the *corresponding strategic-form game* $\bar{G} = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

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For now, let us consider pure strategies only!

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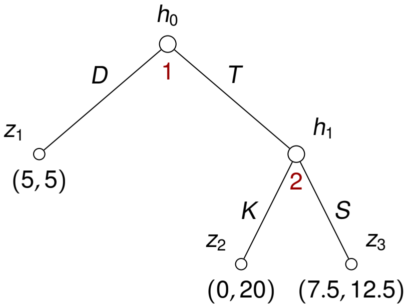

Is any strategy strictly (weakly, very weakly) dominant?

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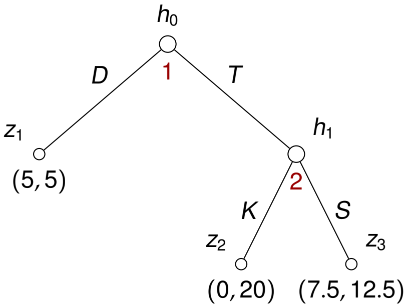

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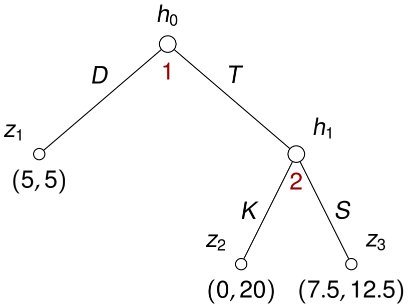

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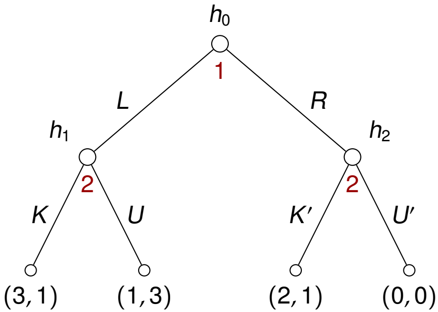

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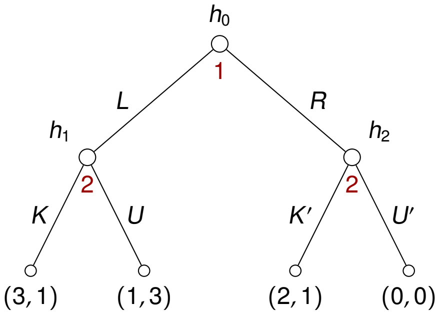

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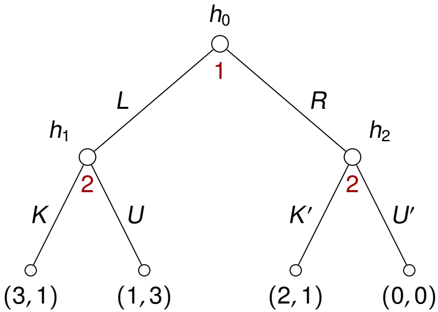

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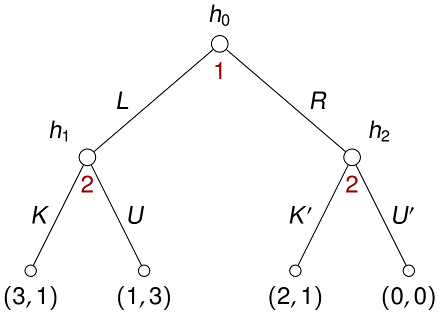

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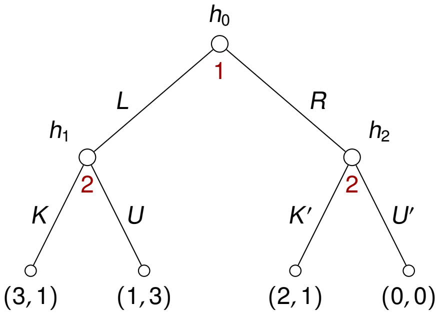

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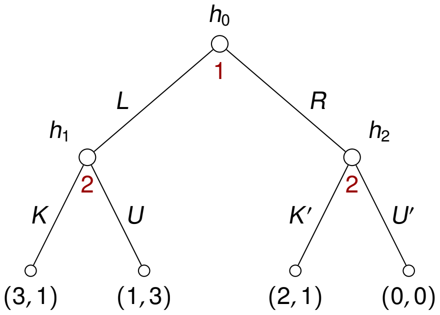

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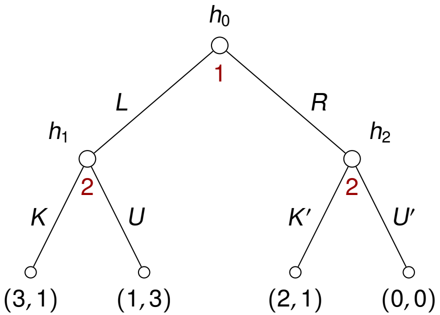

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Two Nash equilibria in pure strategies: (L, UU') and (R, UK')

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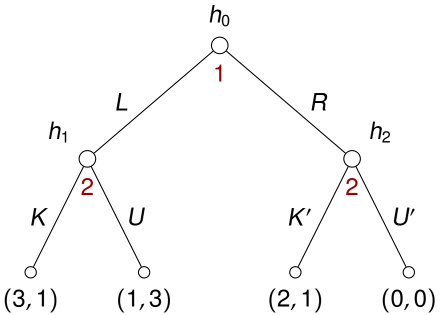

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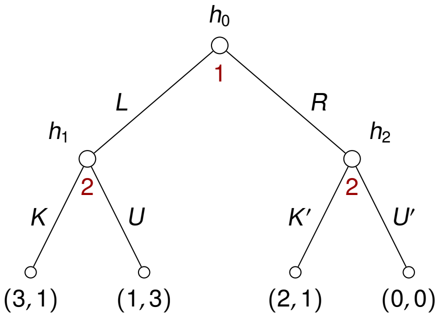

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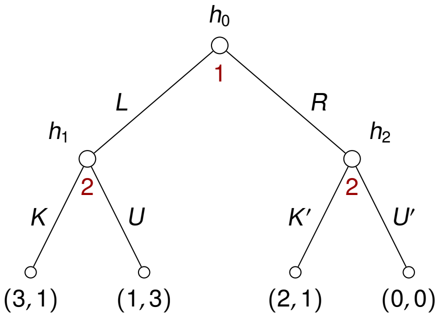

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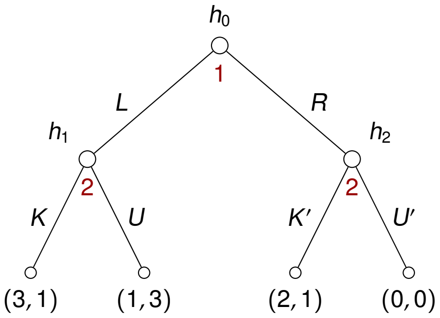

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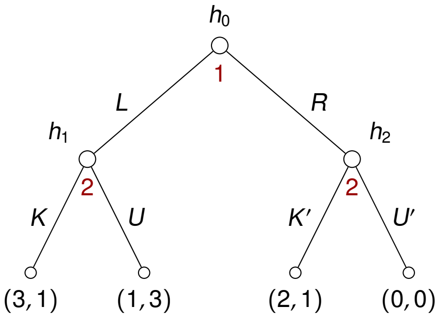

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However, the threat is not *credible*, once a play reaches h_2, a rational player 2 chooses K'.

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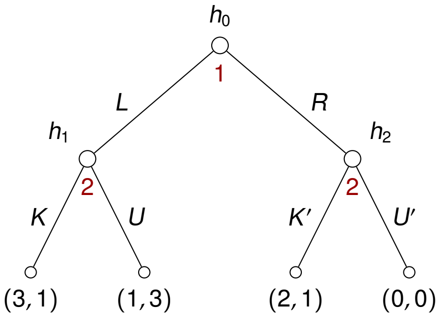

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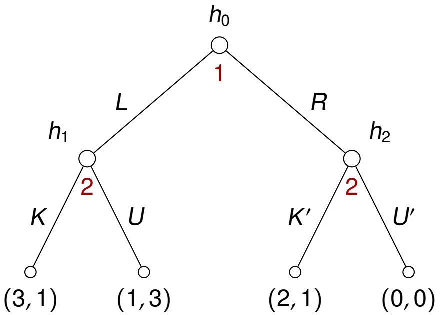

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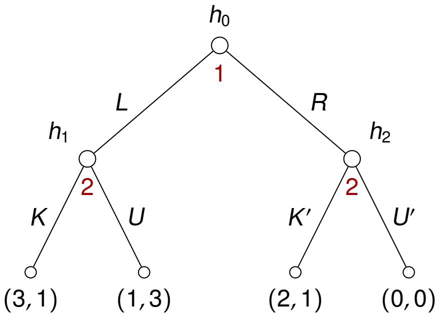

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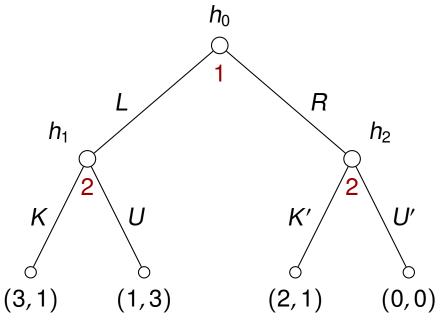

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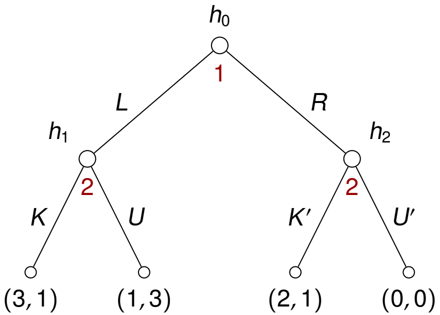

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This equilibrium is called *subgame perfect*.

Subgame Perfect Equilibria

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Definition 44

A *subgame perfect equilibrium (SPE)* in pure strategies is a pure strategy profile $s \in S$ such that for any subgame G^h of G,
the restriction of s to H^h is a Nash equilibrium in pure strategies in G^h.

A restriction of $s = (s_1, \ldots, s_n) \in S$ to H^h is a strategy profile $s^h = (s_1^h, \ldots, s_n^h)$ where $s_i^h(h') = s_i(h')$ for all $i \in N$ and all $h' \in H_i \cap H^h$.

Stackelberg Competition – SPE

- $N = \{1, 2\}$, $A = [0, \infty)$
- $H = \{h_0, h_1^{q_1} \mid q_1 \in [0, \infty)\}$, $Z = \{z^{q_1, q_2} \mid q_1, q_2 \in [0, \infty)$
- $\chi(h_0) = [0, \infty)$, $\chi(h_1^{q_1}) = [0, \infty)$, $\rho(h_0) = 1$, $\rho(h_1^{q_1}) = 2$
- $\pi(h_0, q_1) = h_1^{q_1}$, $\pi(h_1^{q_1}, q_2) = z^{q_1, q_2}$
- The payoffs are $u_1(z^{q_1, q_2}) = q_1(\kappa - c - q_1 - q_2)$, $u_2(z^{q_1, q_2}) = q_2(\kappa - c - q_1 - q_2)$

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Then $u_1(z^{q_1,q_2}) = \theta^2/8$ and $u_2(z^{q_1,q_2}) = \theta^2/16$.

Note that firm 1 has an advantage as a leader.

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An algorithm for computing SPE for finite perfect-information extensive-form games.

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Correctness of Backward Induction

Theorem 45

For every finite perfect-information extensive-form game and for each node h the attached s^h is a SPE and the attached vector $u(h)$ satisfies $u(h) = u(s^h) = (u_1(s^h), \ldots, u_n(s^h))$.

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In both cases the deviation of player i leads to smaller or equal payoff. Apparently, $u(s^h) = u(s^{h_{\max}}) = u(h_{\max}) = u(h)$.

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Recall that in the model of chess, the payoffs were from $\{1, 0, -1\}$ and $u_1 = -u_2$ (i.e. it is zero-sum).

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Question: Which one is the right answer?
Answer: Nobody knows yet ... the tree is too big!
Even with $\sim$ 200 depth & $\sim$ 5 moves per node: 5^{200} nodes!

Efficient Algorithms for Pure Nash Equilibria

In the step 2. of the backward induction, the algorithm may choose *an arbitrary* $h_{\max} \in \operatorname{argmax}_{h' \in K} u_{\rho(h)}(h')$ and always obtain a SPE.

In order to compute all SPE, the algorithm may systematically search through all possible choices of $h_{\max}$ throughout the induction.

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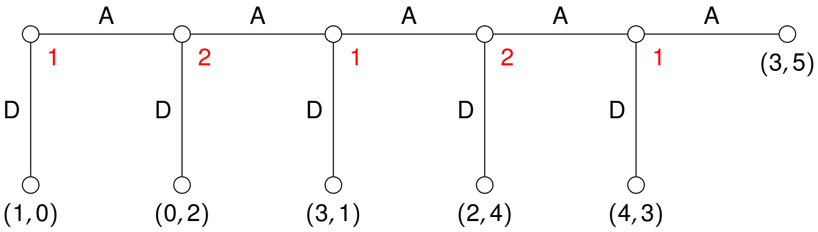
For details, extensions etc. see e.g.

- PB016 Artificial Intelligence I
- Multi-player alpha-beta prunning, R. Korf, *Artificial Intelligence 48*, pages 99-111, 1991
- Artificial Intelligence: A Modern Approach (3rd edition), S. Russell and P. Norvig, *Prentice Hall*, 2009

Example

Centipede game:



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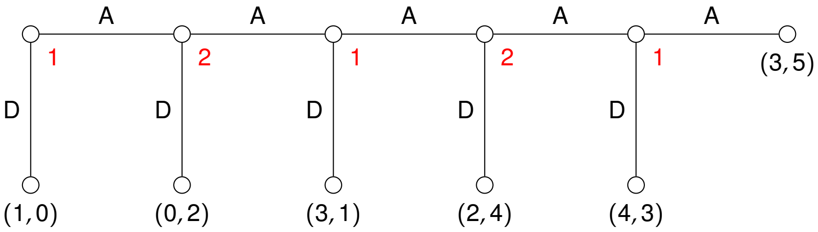

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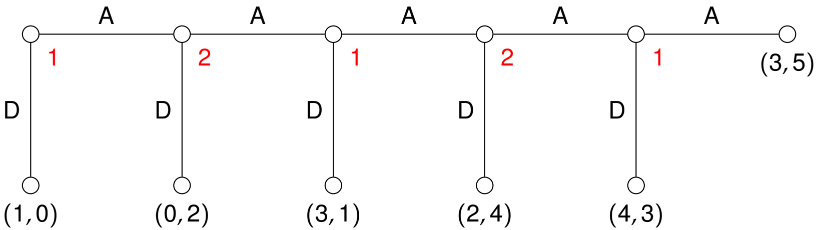

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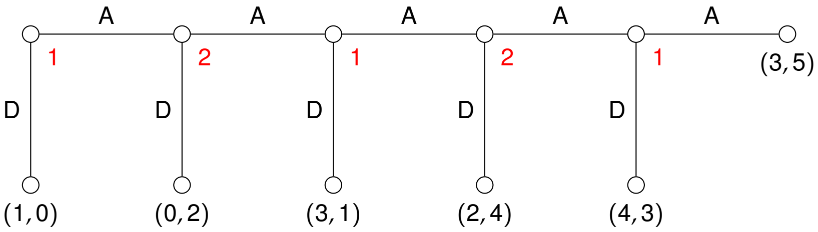

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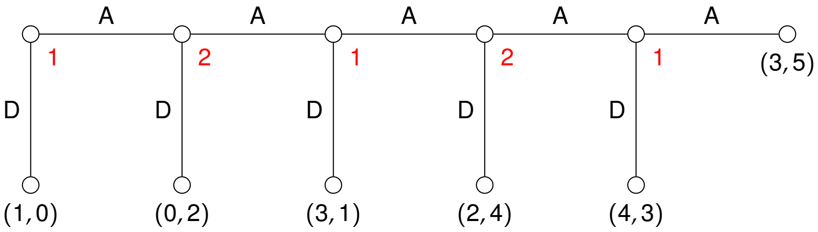

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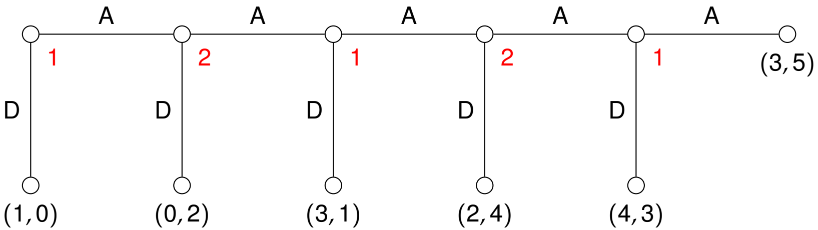

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- In laboratory setting, people usually play A for several steps.
- There is a theoretical problem: Imagine, that you are player 2. What would you do when player 1 chooses A in the first step? The SPE analysis says that you should go down, but the same analysis also says that the situation you are in cannot appear :-)

Dynamic Games of Complete Information

Extensive-Form Games

Mixed and Behavioral Strategies

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Assume two players and a **finite** extensive-form game G.

Definition 46

A *mixed strategy* σ_i of player i in G is a mixed strategy of player i in the corresponding strategic-form game.

I.e., a mixed strategy σ_i of player i in G is a probability distribution on S_i (recall that S_i is the set of all pure strategies, i.e., functions of the form $s_i : H_i \to A$).

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A *behavioral strategy* of player i in G is a function $\beta_i : H_i \to \Delta(A)$ such that for every $h \in H_i$ and every $a \in A$: $\beta_i(h)(a) = 0$ if $a \notin \chi(h)$.

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Given a profile $\beta = (\beta_1, \beta_2)$ of behavioral strategies, we denote by $P_\beta(z)$ the probability of reaching $z \in Z$ when β is used, i.e.,

$$P_\beta(z) = \prod_{\ell=1}^{k} \beta_{\rho(h_{\ell-1})}(h_\ell)(a_\ell)$$

where $h_0 a_1 h_1 a_2 h_2 \cdots a_k h_k$ is the unique path from h_0 to $h_k = z$.

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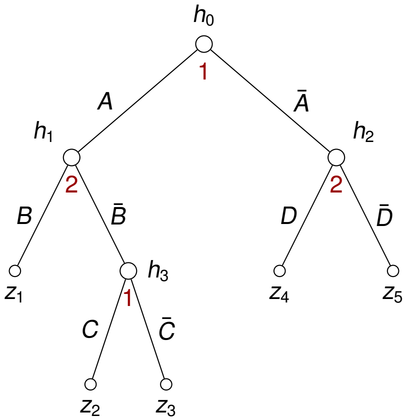
We define $u_i(\beta) := \sum_{z \in Z} P_\beta(z) \cdot u_i(z)$.

Behavioral Strategies: Example



Pure strategies of player 1:

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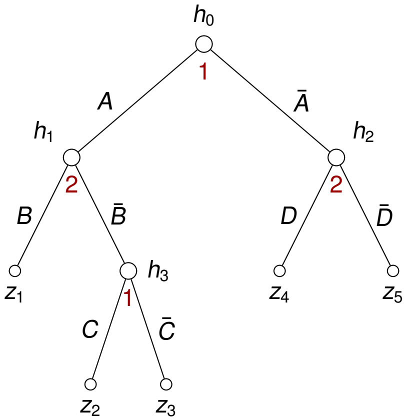
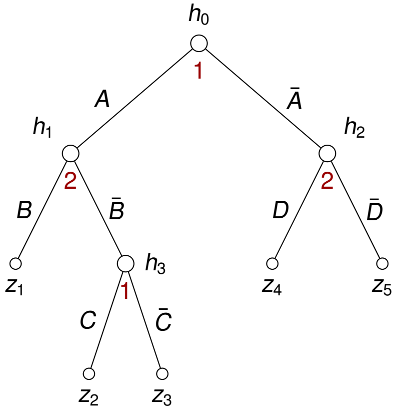


Pure strategies of player 1: AC, $A\bar{C}$, $\bar{A}C$, $\bar{A}\bar{C}$

An example of a mixed strategy σ_1 of player 1:
$\sigma_1(AC) = \frac{1}{3}$, $\sigma_1(A\bar{C}) = \frac{1}{9}$, $\sigma_1(\bar{A}C) = \frac{1}{6}$ and $\sigma_1(\bar{A}\bar{C}) = \frac{11}{18}$

Behavioral Strategies: Example

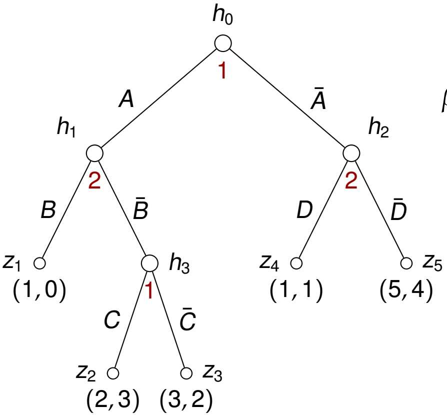


An example of behavioral strategies of both players:

- player 1: $\beta_1(h_0)(A) = \frac{1}{3}$ and $\beta_1(h_3)(C) = \frac{1}{2}$
- player 2: $\beta_2(h_1)(B) = \frac{1}{4}$ and $\beta_2(h_2)(D) = \frac{1}{5}$

$P_{(\beta_1,\beta_2)}(z_2) = \frac{1}{3}\left(1 - \frac{1}{4}\right)\frac{1}{2} = \frac{1}{8}$

Behavioral Strategies: Example

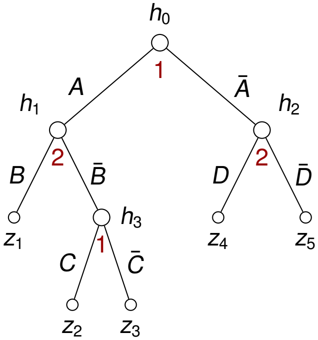


$\beta = (\beta_1, \beta_2)$

- player 1: $\beta_1(h_0)(A) = \frac{1}{3}$ and $\beta_1(h_3)(C) = \frac{1}{2}$
- player 2: $\beta_2(h_1)(B) = \frac{1}{4}$ and $\beta_2(h_2)(D) = \frac{1}{5}$

$$
\begin{aligned}
u_1(\beta) &= P_\beta(z_1) \cdot 1 + P_\beta(z_2) \cdot 2 + P_\beta(z_3) \cdot 3 + P_\beta(z_4) \cdot 1 + P_\beta(z_5) \cdot 5 \\
&= \frac{1}{3}\frac{1}{4}1 + \frac{1}{3}\frac{3}{4}\frac{1}{2}2 + \frac{1}{3}\frac{3}{4}\frac{1}{2}3 + \frac{2}{3}\frac{1}{5}1 + \frac{2}{3}\frac{4}{5}5 \approx 3.508
\end{aligned}
$$

Pure Strategies as Behavioral



Each pure strategy can be seen as a behavioral strategy. Consider e.g. $s_1 : H_1 \to A$ defined by $s_1(h_0) = A$ and $s_1(h_3) = C$.

The corresponding behavioral strategy β_1 would satisfy $\beta_1(h_0)(A) = \beta_1(h_3)(C) = 1$ (i.e. select actions chosen by s_1 with prob. 1).

Now given a behavioral strategy β_2 of player 2 defined by $\beta_2(h_1)(B) = \frac{1}{4}$ and $\beta_2(h_2)(D) = \frac{1}{5}$ we obtain

$$P_{(s_1,\beta_2)}(z_2) = P_{(\beta_1,\beta_2)}(z_2) = 1\left(1 - \frac{1}{4}\right)1 = \frac{3}{4}$$

Mixed/Behavioral Profiles

Let $\alpha = (\alpha_1, \alpha_2)$ be a strategy profile where each α_i is either mixed or behavioral.

The game is played as follows:

- If α_1 mixed, select randomly a pure strategy β_1 according to α_1, else $\beta_1 := \alpha_1$.
- If α_2 mixed, select randomly a pure strategy β_2 according to α_2, else $\beta_2 := \alpha_2$.
- Play (β_1, β_2) and collect payoffs.

Denote the resulting payoffs by $u_1(\alpha)$ and $u_2(\alpha)$.

Lemma 48
For every mixed/behavioral strategy α_1 of player 1 there is a behavioral/mixed strategy α'_1 such that for every mixed/behavioral strategy α_2 we have that $u_i(\alpha_1, \alpha_2) = u_i(\alpha'_1, \alpha_2)$ for $i \in \{1, 2\}$.

Dynamic Games of Complete Information

Extensive-Form Games

Imperfect-Information Games

Extensive-form of Matching Pennies

Is it possible to model Matching pennies using extensive-form games?

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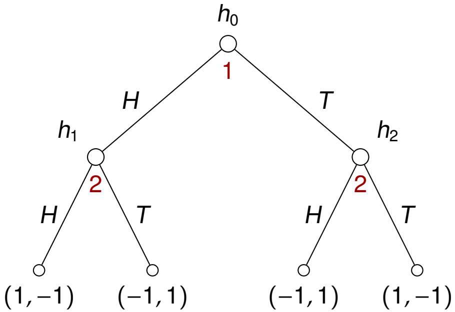
	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$



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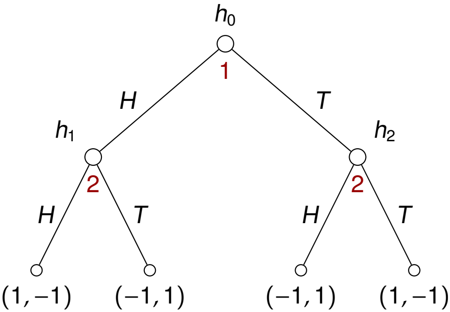

The problem is that player 2 is "perfectly" informed about the choice of player 1. In particular, there are pure Nash equilibria (H, TH) and (T, TH) in the extensive-form game as opposed to the strategic-form.

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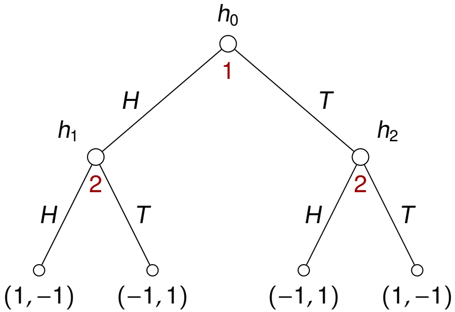

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Reversing the order of players does not help.

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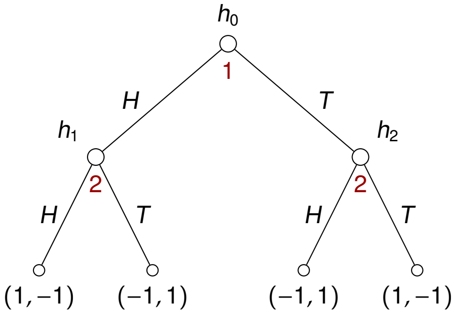

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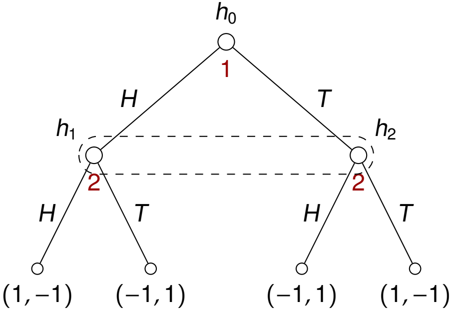
Reversing the order of players does not help.

We need to extend the formalism to be able to hide some information about previous moves.

Extensive-form of Matching Pennies

Matching pennies can be modeled using an *imperfect-information* extensive-form game:



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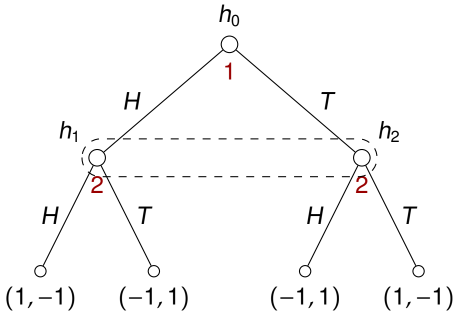

Here h_1 and h_2 belong to the same *information set* of player 2.

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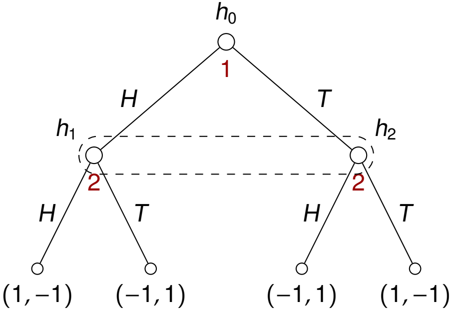

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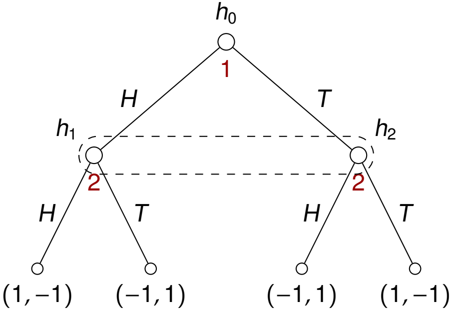

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So even though players do not move simultaneously, the information player 2 has about the current situation is the same as in the simultaneous case.

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Given $h \in H$, we denote by $I(h)$ the information set $I_{i,j}$ containing h.

Given an information set $I_{i,j}$, we denote by $\chi(I_{i,j})$ the set of all actions enabled in some (and hence all) nodes of $I_{i,j}$.

Imperfect Information Games – Strategies

Now we define the set of pure, mixed, and behavioral strategies in G_{imp} as subsets of pure, mixed, and behavioral strategies, resp., in G_{perf} that respect the information sets.

Let $G_{imp} = (G_{perf}, I)$ be an imperfect-information extensive-form game where $G_{perf} = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$.

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Definition 49

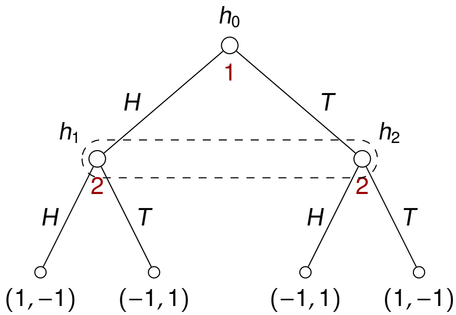
A *pure strategy* of player i in G_{imp} is a pure strategy s_i in G_{perf} such that for all $j = 1, \ldots, k_i$ and all $h, h' \in I_{i,j}$ holds $s_i(h) = s_i(h')$.

Note that each s_i can also be seen as a function $s_i : I_i \to A$ such that for every $I_{i,j} \in I_i$ we have that $s_i(I_{i,j}) \in \chi(I_{i,j})$.

As before, we denote by S_i the set of all pure strategies of player i in G_{imp}, and by $S = S_1 \times \cdots \times S_n$ the set of all pure strategy profiles.

As in the perfect-information case we have a corresponding strategic-form game $\bar{G}_{imp} = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$.

Matching Pennies



$I_1 = \{I_{1,1}\}$ where $I_{1,1} = \{h_0\}$

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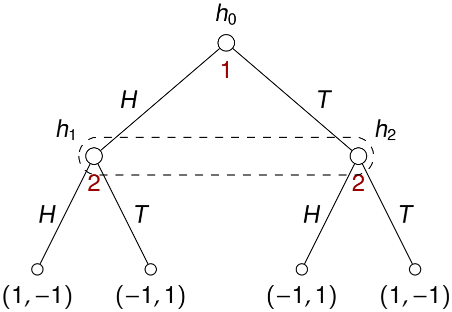

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Example of pure strategies:

- $s_1(I_{1,1}) = H$ which describes the strategy $s_1(h_0) = H$
- $s_2(I_{2,1}) = T$ which describes the strategy $s_2(h_1) = s_2(h_2) = T$
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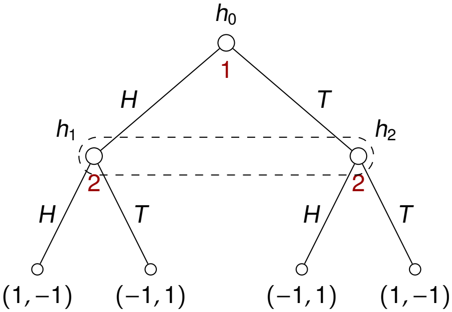

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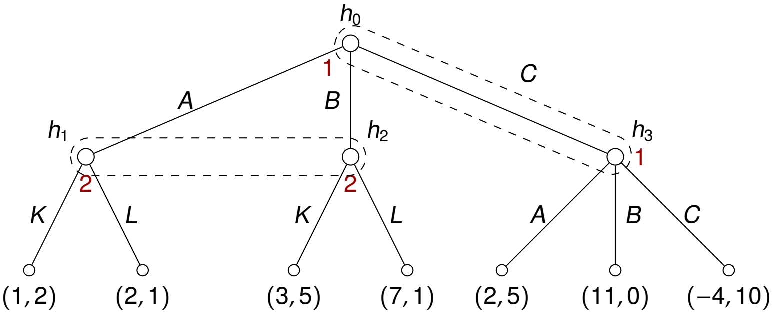
So we really have strategies H, T for player 1 and H, T for player 2.

Weird Example

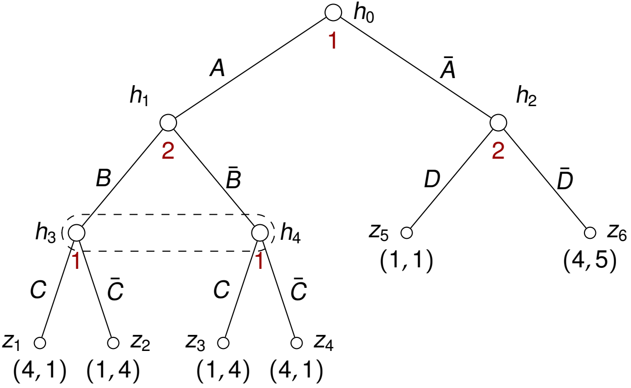


Note that $I_1 = \{I_{1,1}\}$ where $I_{1,1} = \{h_0, h_3\}$

and that $I_2 = \{I_{2,1}\}$ where $I_{2,1} = \{h_1, h_2\}$

What pure strategies are in this example?

SPE with Imperfect Information



What we designate as subgames to allow the backward induction?

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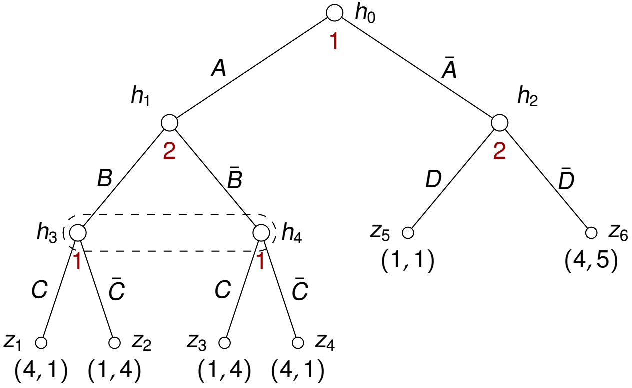

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Only subtrees rooted in h_1, h_2, and h_0 (together with all subtrees rooted in terminal nodes)

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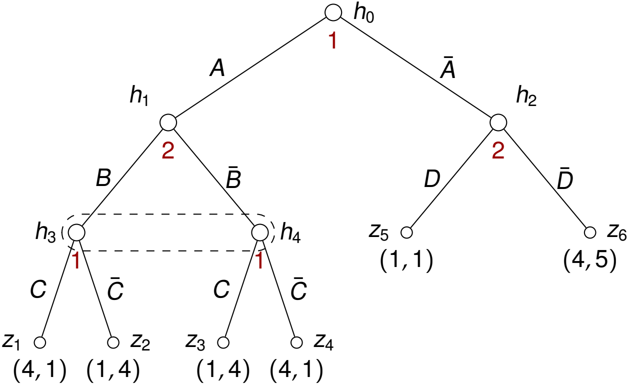

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Note that subtrees rooted in h_3 and h_4 cannot be considered as "independent" subgames because their individual solutions cannot be combined to a single best response in the information set $\{h_3, h_4\}$.

SPE with Imperfect Information

Let $G_{imp} = (G_{perf}, I)$ be an imperfect-information extensive-form game where $G_{perf} = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$ is the underlying perfect-information extensive-form game.

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Let us denote by H_{proper} the set of all $h \in H$ that satisfy the following: For every h' reachable from h, we have that either all nodes of $I(h')$ are reachable from h, or no node of $I(h')$ is reachable from h.
Intuitively, $h \in H_{proper}$ iff every information set $I_{i,j}$ is either completely contained in the subtree rooted in h, or no node of $I_{i,j}$ is contained in the subtree.

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Definition 50

For every $h \in H_{proper}$ we define a subgame G^h_{imp} to be the imperfect information game (G^h_{perf}, I^h) where I^h is the restriction of I to H^h.

Note that as subgames of G_{imp} we consider only subgames of G_{perf} that respect the information sets, i.e., are rooted in nodes of H_{proper}.

SPE with Imperfect Information

Let $G_{imp} = (G_{perf}, I)$ be an imperfect-information extensive-form game where $G_{perf} = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$ is the underlying perfect-information extensive-form game.

Let us denote by H_{proper} the set of all $h \in H$ that satisfy the following: For every h' reachable from h, we have that either all nodes of $I(h')$ are reachable from h, or no node of $I(h')$ is reachable from h.
Intuitively, $h \in H_{proper}$ iff every information set $I_{i,j}$ is either completely contained in the subtree rooted in h, or no node of $I_{i,j}$ is contained in the subtree.

Definition 50
For every $h \in H_{proper}$ we define a subgame G^h_{imp} to be the imperfect information game (G^h_{perf}, I^h) where I^h is the restriction of I to H^h.
Note that as subgames of G_{imp} we consider only subgames of G_{perf} that respect the information sets, i.e., are rooted in nodes of H_{proper}.

Definition 51
A strategy profile $s \in S$ is a subgame perfect equilibrium (SPE) if s^h is a Nash equilibrium in every subgame G^h_{imp} of G_{imp} (here $h \in H_{proper}$).

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2. Starting with terminal nodes, the labeling proceeds bottom up. Terminal nodes are labeled similarly as in the perfect-inf. case.
3. Consider $h \in H_{proper}$, let K be the set of all $h' \in \left(H_{proper} \cup Z\right) \smallsetminus \{h\}$ that are h's closest descendants out of $H_{proper} \cup Z$.

   I.e., $h' \in K$ iff $h' \neq h$ is reachable from h and the unique path from h to h' visits only nodes of $\mathcal{H} \smallsetminus H_{proper}$ (except the first and the last node).

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   For every $h' \in K$ we have already computed a SPE $s^{h'}$ in $G^{h'}_{imp}$ and the vector of corresponding payoffs $u(h')$.

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   I.e., $h' \in K$ iff $h' \neq h$ is reachable from h and the unique path from h to h' visits only nodes of $\mathcal{H} \setminus H_{proper}$ (except the first and the last node).

   For every $h' \in K$ we have already computed a SPE $s^{h'}$ in $G_{imp}^{h'}$ and the vector of corresponding payoffs $u(h')$.
4. Now consider all nodes of K as terminal nodes where each $h' \in K$ has payoffs $u(h')$. This gives a new game in which we compute an equilibrium $\bar{s}^h$ together with the vector $u(h)$.

   The equilibrium s^h is then obtained by "concatenating" $\bar{s}^h$ with all $s^{h'}$, here $h' \in K$, in the subgames $G_{imp}^{h'}$ of G_{imp}^h.

Mutually Assured Destruction

Analysis of Cuban missile crisis of 1962
(as described in *Games for Business and Economics* by R. Gardner)

- The crisis started with United States' discovery of Soviet nuclear missiles in Cuba.
- The USSR then backed down, agreeing to remove the missiles from Cuba, which suggests that US had a credible threat "if you don't back off we both pay dearly".

Question: Could this indeed be a credible threat?

Mutually Assured Destruction (Cont.)

Model as an extensive-form game:

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- First, player 1 (US) chooses to either ignore the incident (I), resulting in maintenance of status quo (payoffs $(0, 0)$), or escalate the situation (E).

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- Following escalation by player 1, player 2 can back down (B), causing it to lose face (payoffs $(10, -10)$), or it can choose to proceed to a nuclear confrontation (N).

Mutually Assured Destruction (Cont.)

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  - If both retreat, the payoffs are $(-5, -5)$, a small loss due to a mobilization process.

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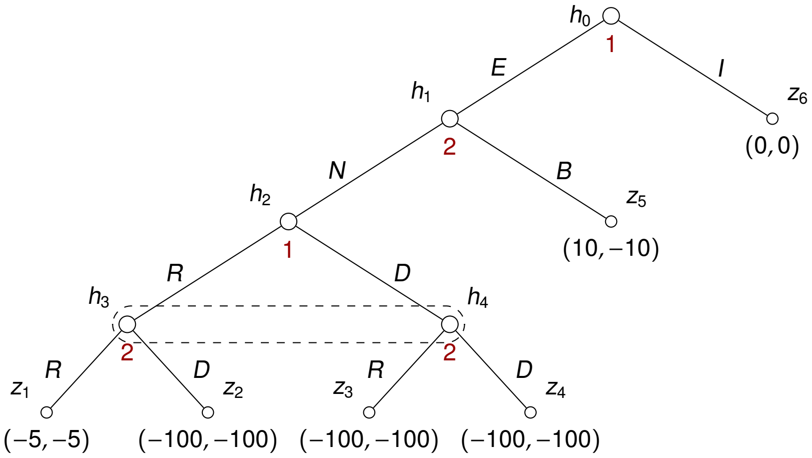
- Upon this choice, the players play a simultaneous-move game in which they can either retreat (R), or choose doomsday (D).
  - If both retreat, the payoffs are $(-5, -5)$, a small loss due to a mobilization process.
  - If either of them chooses doomsday, then the world destructs and payoffs are $(-100, -100)$.

Find SPE in pure strategies.

Mutually Assured Destruction (Cont.)



Solve $G^{h_2}_{imp}$ (a strategic-form game). Then $G^{h_1}_{imp}$ by solving a game rooted in h_1 with terminal nodes h_2, z_5 (payoffs in h_2 correspond to an equilibrium in $G^{h_2}_{imp}$). Finally solve G_{imp} by solving a game rooted in h_0 with terminal nodes h_1, z_6 (payoffs in h_1 have been computed in the previous step).

Mixed and Behavioral Strategies

Definition 52

A *mixed strategy* σ_i of player i in G_{imp} is a mixed strategy of player i in the corresponding strategic-form game $\bar{G}_{imp} = (N, (S_i)_{i \in N}, u_i)$.

Do not forget that now $s_i \in S_i$ iff s_i is a pure strategy that assigns the same action to all nodes of every information set. Hence each $s_i \in S_i$ can be seen as a function $s_i : I_i \to A$.

As before, we denote by Σ_i the set of all mixed strategies of player i.

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As before, we denote by Σ_i the set of all mixed strategies of player i.

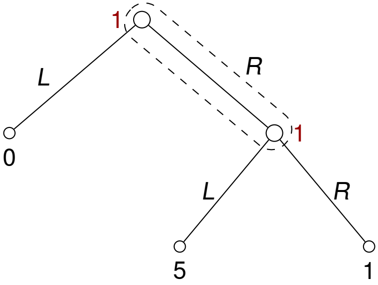
Definition 53

A *behavioral strategy* of player i in G_{imp} is a behavioral strategy β_i in G_{perf} such that for all $j = 1, \ldots, k_i$ and all $h, h' \in I_{i,j} : \beta_i(h) = \beta_i(h')$.

Each β_i can be seen as a function $\beta_i : I_i \to \Delta(A)$ such that for all $I_{i,j} \in I_i$ we have $supp(\beta_i(I_{i,j})) \subseteq \chi(I_{i,j})$.

Are they equivalent as in the perfect-information case?

Example: Absent Minded Driver



Only one player: A driver who has to take a turn at a particular junction. There are two identical junctions, the first one leads to a wrong neighborhood where the driver gets completely lost (payoff 0), the second one leads home (payoff 5). If the driver misses both, there is a longer way home (payoff 1). The problem is that after missing the first turn, the driver forgets that he missed the turn.

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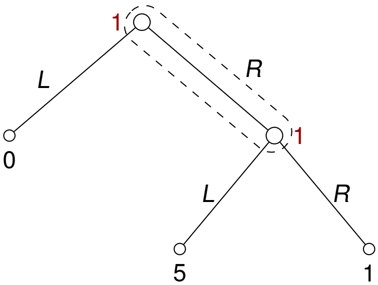

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Behavioral strategy: $\beta_1(I_{1,1})(L) = \frac{1}{2}$ has the expected payoff $\frac{3}{2}$.

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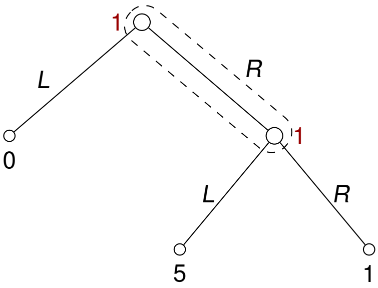

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Behavioral strategy: $\beta_1(I_{1,1})(L) = \frac{1}{2}$ has the expected payoff $\frac{3}{2}$.

No mixed strategy gives a larger payoff than 1 since no pure strategy ever reaches the terminal node with payoff 5.

Kuhn's Theorem

Player i has *perfect recall* in G_{imp} if the following holds:

- Every information set of player i (i.e., *his own*) intersects every path from the root h_0 to a terminal node at most once.
- Every two paths from the root that end in the same information set of player i
  - pass through the same information sets of player i,
  - and in the same order,
  - and in every such information set the two paths choose the same action.

  May, however, pass through *different* information sets of other players and other players may choose different actions along each of the paths!

  I.e. each information set J of player i determines the sequence of information sets of player i and actions taken by player i along any path reaching J.

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Theorem 54 (Kuhn, 1953)

Assuming perfect recall, every mixed strategy can be translated to a behavioral strategy (and vice versa) so that the payoff for the resulting strategy is the same in any mixed profile.

Dynamic Games of Complete Information

Repeated Games

Finitely Repeated Games

Example – repeated prisoner's dilemma

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Imagine that the criminals are being arrested repeatedly.

Can they somewhat reflect upon their experience in order to play "better"?

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In what follows, we consider strategic-form games played repeatedly

- for finitely many rounds, the final payoff of each player will be the average of payoffs from all rounds
- for infinitely many rounds we consider the long-run average payoff

We analyze Nash equilibria and sub-game perfect equilibria.

We stick with pure strategies only!

Finitely Repeated Games

Let $G = (\{1,2\}, (S_1, S_2), (u_1, u_2))$ be a finite strategic-form game of two players.

A *T-stage game $G_{T\text{-rep}}$ based on G* proceeds in T *stages* so that in a stage $t \geq 1$, players choose a strategy profile $s^t = (s_1^t, s_2^t)$.

After T stages, both players collect the average payoff $\sum_{t=1}^{T} u_i(s^t) \,/\, T$.

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After T stages, both players collect the average payoff $\sum_{t=1}^{T} u_i(s^t) \,/\, T$.

A *history of length $0 \leq t \leq T$* is a sequence $h = s^1 \cdots s^t \in S^t$ of t strategy profiles. Denote by $H(t)$ the set of all histories of length t.

A *pure strategy* for player i in a T-stage game $G_{T\text{-}rep}$ is a function

$$\tau_i : \bigcup_{t=0}^{T-1} H(t) \rightarrow S_i$$

which for every possible history chooses a next step for player i.

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which for every possible history chooses a next step for player i.

Every strategy profile $\tau = (\tau_1, \tau_2)$ in $G_{T\text{-}rep}$ induces a sequence of pure strategy profiles $w_\tau = s^1 \cdots s^T$ in G so that $s_i^t = \tau_i(s^1 \cdots s^{t-1})$.

Given a pure strategy profile τ in $G_{T\text{-}rep}$ such that $w_\tau = s^1 \cdots s^T$, define the payoffs $u_i(\tau) = \sum_{t=1}^{T} u_i(s^t) \,/\, T$.

Example

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Consider a 3-stage game.

Examples of histories: ϵ, (C, S), $(C, S)(S, S)$, $(C, S)(S, S)(C, C)$

Here the last one is terminal, obtained using τ_1, τ_2 s.t.:

$\tau_1(\epsilon) = C$, $\tau_1((C, S)) = S$, $\tau_1((C, S)(S, S)) = C$
$\tau_2(\epsilon) = S$, $\tau_2((C, S)) = S$, $\tau_2((C, S)(S, S)) = C$

Thus $w_{(\tau_1, \tau_2)} = (C, S)(S, S)(C, C)$

$u_1(\tau_1, \tau_2) = (0 + (-1) + (-5))/3 = -2$
$u_2(\tau_1, \tau_2) = (-20 + (-1) + (-5))/3 = -26/3$

Note that the above definitions of τ_1 and τ_2 are incomplete! To define the strategies completely, we have to define their values on *all* histories including the ones that do not appear along the play played according to them.

Finitely Repeated Games in Extensive-Form

Every T-stage game $G_{T\text{-}rep}$ can be defined as an imperfect information extensive-form game.

Define an imperfect-information extensive-form game $G^{rep}_{imp} = (G^{rep}_{perf}, I)$ such that $G^{rep}_{perf} = (\{1,2\}, A, H, Z, \chi, \rho, \pi, h_0, u)$ where

- $A = S_1 \cup S_2$
- $H = (S_1 \times S_2)^{\leq T} \cup (S_1 \times S_2)^{<T} \cdot S_1$

  Intuitively, elements of $(S_1 \times S_2)^{\leq k}$ are possible histories; $(S_1 \times S_2)^{<k} \cdot S_1$ is used to simulate a simultaneous play of G by letting player 1 choose first and player 2 second.
- $Z = (S_1 \times S_2)^T$
- $\chi(\epsilon) = S_1$ and $\chi(h \cdot s_1) = S_2$ for $s_1 \in S_1$, and $\chi(h \cdot (s_1, s_2)) = S_1$ for $(s_1, s_2) \in S$
- $\rho(\epsilon) = 1$ and $\rho(h \cdot s_1) = 2$ and $\rho(h \cdot (s_1, s_2)) = 1$
- $\pi(\epsilon, s_1) = s_1$ and $\pi(h \cdot s_1, s_2) = h \cdot (s_1, s_2)$ and $\pi(h \cdot (s_1, s_2), s_1') = h \cdot (s_1, s_2) \cdot s_1'$
- $h_0 = \epsilon$ and $u_i((s_1^1, s_2^1)(s_1^2, s_2^2) \cdots (s_1^T, s_2^T)) = \sum_{t=1}^T u_i(s_1^t, s_2^t) \,/\, T$

Finitely Repeated Games in Extensive-Form

The set of information sets is defined as follows: Let $h \in H_1$ be a node of player 1, then

- there is exactly one information set of player 1 containing h as the only element,
- there is exactly one information set of player 2 containing all nodes of the form $h \cdot s_1$ where $s_1 \in S_1$.

Intuitively, in every round, player 1 has a complete information about results of past plays,

player 1 chooses a pure strategy $s_1 \in S_1$,

player 2 is *not* informed about s_1 but still has a complete information about results of all previous rounds,

player 2 chooses a pure strategy $s_2 \in S_2$ and both players are informed about the result.

Finitely Repeated Games – Equilibria

Definition 55

A strategy profile $\tau = (\tau_1, \tau_2)$ in a T-stage game $G_{T\text{-}rep}$ is a Nash equilibrium if for every $i \in \{1, 2\}$ and every τ_i' we have

$$u_i(\tau_1, \tau_2) \geq u_i(\tau_i', \tau_{-i})$$

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To define SPE we use the following notation. Given a history $h = s^1 \cdots s^t$ and a strategy τ_i of player i, we define a strategy τ_i^h in $(T - t)$-stage game based on G by

$$\tau_i^h(\bar{s}^1 \cdots \bar{s}^{\bar{t}}) = \tau_i(s^1 \cdots s^t \bar{s}^1 \cdots \bar{s}^{\bar{t}}) \quad \text{for every sequence } \bar{s}^1 \cdots \bar{s}^{\bar{t}}$$

(i.e. τ_i^h behaves as τ_i after h)

Definition 56

A strategy profile $\tau = (\tau_1, \tau_2)$ in a T-stage game $G_{T\text{-}rep}$ is a subgame-perfect Nash equilibrium (SPE) if for every history h the profile (τ_1^h, τ_2^h) is a Nash equilibrium in the $(T - |h|)$-stage game based on G.

SPE with Single NE in G

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Consider a T-stage game based on Prisoner's dilemma.

For every T, find a SPE.

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... there is one, play (C, C) all the time. Is it all?

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For every T, find a SPE.

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Theorem 57
Let G be an arbitrary finite strategic-form game. If G has a unique Nash equilibrium, then playing this equilibrium all the time is the unique SPE in the T-stage game based on G.

Proof.
By backward induction, players have to play the NE in the last stage. As the behavior in the last stage does not depend on the behavior in the $(T-1)$-th stage, they have to play the NE also in the $(T-1)$-th stage. Then the same holds in the $(T-2)$-th stage, etc. □

Further Discussion of Prisoner's Dilemma

	C	S
C	$-5, -5$	$0, -20$
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Are there other NE (that are not SPE) in the repeated Prisoner's dilemma?

Further Discussion of Prisoner's Dilemma

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Are there other NE (that are not SPE) in the repeated Prisoner's dilemma?

To simplify our discussion, we use the following notation: $X{-}YZ$, where $X, Y, Z \in \{C, S\}$ denotes the following strategy:

- In the first phase, play X
- In the second phase, play Y if the opponent plays C in the first phase, otherwise play Z

There are 4 NE: These are the four profiles that lead to $(C, C)(C, C)$, i.e., each player plays either $C{-}CC$, or $C{-}CS$.

Further Discussion of Prisoner's Dilemma

	C	S
C	$-5, -5$	$0, -20$
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The strategy C strictly dominates S in the Prisoner's dilemma.

Is there a strictly dominant strategy in the 2-stage game based on the Prisoner's dilemma?

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The strategy C strictly dominates S in the Prisoner's dilemma.

Is there a strictly dominant strategy in the 2-stage game based on the Prisoner's dilemma?

If player 2 plays $S{-}CS$, then the best responses of player 1 are $S{-}CC$ and $S{-}SC$.
(The strategy $S{-}CS$ is usually called "tit-for-tat".)

If player 2 plays $S{-}SC$, then the best responses are $C{-}SC$ and $C{-}CC$.

So there is no strictly dominant strategy for player 1.
(Which would be among the best responses for all strategies of player 2.)

SPE with Multiple NE in G

Let $s = (s_1, s_2)$ be a Nash equilibrium in G.

Define a strategy profile $\tau = (\tau_1, \tau_2)$ in $G_{T\text{-}rep}$ where

- τ_1 chooses s_1 in every stage
- τ_2 chooses s_2 in every stage

Proposition 3

τ is a SPE in $G_{T\text{-}rep}$ for every $T \geq 1$.

SPE with Multiple NE in G

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Proposition 3

τ is a SPE in $G_{T\text{-}rep}$ for every $T \geq 1$.

Proof.

Apparently, changing τ_i in some stage(s) may only result in the same or worse payoff for player i, since the other player always plays s_2 independent of the choices of player 1. □

The proposition may be generalized by allowing players to play different equilibria in particular stages

I.e., consider a sequence of NE $s^1, s^2, \dots, s^T$ in G and assume that in stage ℓ player i plays s_i^ℓ

Does this cover all possible SPE in finitely repeated games?

SPE with Multiple NE in G

	m	f	r
M	$4, 4$	$-1, 5$	$0, 0$
F	$5, -1$	$1, 1$	$0, 0$
R	$0, 0$	$0, 0$	$3, 3$

NE in the above game G : (F, f) and (R, r)

Consider 2-stage game $G_{2\text{-}rep}$ and strategies τ_1, τ_2 where

- τ_1 : Chooses M in stage 1. In stage 2 plays R if (M, m) was played in the first stage, and plays F otherwise.
- τ_2 : Chooses m in stage 1. In stage 2 plays r if (M, m) was played in the first stage, and plays f otherwise.

Is this SPE?

SPE with Multiple NE in G

	m	f	r
M	$4,4$	$-1,5$	$0,0$
F	$5,-1$	$1,1$	$0,0$
R	$0,0$	$0,0$	$3,3$

NE in the above game G : (F, f) and (R, r)

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- τ_2 : Chooses m in stage 1. In stage 2 plays r if (M, m) was played in the first stage, and plays f otherwise.

Is this SPE?

Note that here the players **do not** play a NE in the first step.

The idea is that both players agree to play a Pareto optimal profile. If both comply, then a favorable NE is played in the second stage. If one of them betrays then a "punishing" NE is played.

Dynamic Games of Complete Information

Repeated Games

Infinitely Repeated Games

Infinitely Repeated Games

Let $G = (\{1,2\}, (S_1, S_2), (u_1, u_2))$ be a strategic-form game of two players.

An *infinitely repeated game* G_{irep} based on G proceeds in *stages* so that in each stage, say t, players choose a strategy profile $s^t = (s_1^t, s_2^t)$.

Infinitely Repeated Games

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An *infinitely repeated game G_{irep}* based on G proceeds in *stages* so that in each stage, say t, players choose a strategy profile $s^t = (s_1^t, s_2^t)$.

Recall that a *history of length $t \geq 0$* is a sequence $h = s^1 \cdots s^t \in S^t$ of t strategy profiles. Denote by $H(t)$ the set of all histories of length t.

A *pure strategy* for player i in the infinitely repeated game G_{irep} is a function

$$\tau_i : \bigcup_{t=0}^{\infty} H(t) \to S_i$$

which for every possible history chooses a next step for player i.

Infinitely Repeated Games

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which for every possible history chooses a next step for player i.

Every pure strategy profile $\tau = (\tau_1, \tau_2)$ in G_{irep} induces a sequence of pure strategy profiles $w_\tau = s^1 s^2 \cdots$ in G so that $s^t_i = \tau_i(s^1 \cdots s^{t-1})$.
(Here for $t = 0$ we have that $s^1 \cdots s^{t-1} = \epsilon$.)

Infinitely Repeated Games & Average Payoff

In what follows we assume that all payoffs in the game G are positive and that S is finite!

Let $\tau = (\tau_1, \tau_2)$ be a strategy profile in the infinitely repeated game G_{irep} such that $w_\tau = s^1 s^2 \cdots$.

Definition 58

We define a *long-run average payoff* for player i by

$$u_i^{avg}(\tau) = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} u_i(s^t)$$

(Here $\limsup$ is necessary because τ_i may cause non-existence of the limit.)

The lon-run average payoff $u_i^{avg}(\tau)$ is *well-defined* if the limit $u_i^{avg}(\tau) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} u_i(s^t)$ exists.

Given a strategic-form game G, we denote by G_{irep}^{avg} the infinitely repeated game based on G together with the long-run average payoff.

Definition 59
A strategy profile τ is a Nash equilibrium if $u_i^{avg}(\tau)$ is well-defined for all $i \in N$, and for every i and every τ_i' we have that

$$u_i^{avg}(\tau_i, \tau_{-i}) \quad \geq \quad u_i^{avg}(\tau_i', \tau_{-i})$$

(Note that we demand existence of the defining limit of $u_i^{avg}(\tau_i, \tau_{-i})$ but the limit does not have to exist for $u_i^{avg}(\tau_i', \tau_{-i})$.)

Moreover, $\tau = (\tau_1, \tau_2)$ is a SPE in G_{irep}^{avg} if for every history h we have that (τ_1^h, τ_2^h) is a Nash equilibrium.

Example

Consider the infinitely repeated game based on Prisoner's dilemma:

	C	S
C	$-5,-5$	$0,-20$
S	$-20,0$	$-1,-1$

The grim trigger profile (τ_1, τ_2) where

$$\tau_i(s^1 \cdots s^T) = \begin{cases} S & T = 0 \\ S & s^\ell = (S,S) \text{ for all } 1 \leq \ell \leq T \\ C & \text{otherwise} \end{cases}$$

is a SPE which gives the long-run average payoff -1 to each player.

The intuition behind the grim trigger works as follows: Whenever a player i deviates, the player $-i$ starts playing C for which the best response of player i is also C. So we obtain $(S,S)\cdots(S,S)(X,Y)(C,C)(C,C)\cdots$ (here (X,Y) is either (C,S) or (S,C) depending on who deviates). Apparently, the long-run average payoff is -5 for both players, which is worse than -1.

Example

Consider the infinitely repeated game based on the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

However, other payoffs can be supported by NE. Consider e.g. a strategy profile (τ_1, τ_2) such that

- Both players **cyclically** play as follows:
  - 9 times (S, S)
  - once (S, C)
- If one of the players deviates, then, from that moment on, both play (C, C) forever.

Then (τ_1, τ_2) is also SPE.

Example

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- If one of the players deviates, then, from that moment on, both play (C, C) forever.

Then (τ_1, τ_2) is also SPE.

Apparently, $u_1^{avg}(\tau_1, \tau_2) = \frac{9}{10} \cdot (-1) + (-20)/10 = -29/10$ and $u_1^{avg}(\tau_1, \tau_2) = \frac{9}{10}(-1) = -9/10$.

Player 2 gets a better payoff than from the "best" profile (S, S)!

Outline of the Folk Theorems

The previous examples suggest that other (possibly all?) convex combinations of payoffs may be obtained by means of Nash equilibria.

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No author is listed since these theorems had been known in games community long before they were formalized.

Outline of the Folk Theorems

The previous examples suggest that other (possibly all?) convex combinations of payoffs may be obtained by means of Nash equilibria.

This observation forms a basis for a bunch of theorems, collectively called Folk Theorems.
No author is listed since these theorems had been known in games community long before they were formalized.

In what follows we prove several versions of Folk Theorem concerning achievable payoffs for repeated games.

We consider the following variants:

- Long-run average payoffs & SPE
- Long-run average payoffs & Nash equilibria

Note that similar theorems can be proved also for the discounted payoff.

Folk Theorems – Feasible Payoffs

Definition 60

We say that a vector of payoffs $v = (v_1, v_2) \in \mathbb{R}^2$ is *feasible* if it is a convex combination of payoffs for pure strategy profiles in G with rational coefficients, i.e., if there are rational numbers β_s, here $s \in S$, satisfying $\beta_s \geq 0$ and $\sum_{s \in S} \beta_s = 1$ such that for both $i \in \{1, 2\}$ holds

$$v_i = \sum_{s \in S} \beta_s \cdot u_i(s)$$

We assume that there is $m \in \mathbb{N}$ such that each β_s can be written in the form $\beta_s = \gamma_s / m$.

The following theorems can be extended to a notion of feasible payoffs using *arbitrary, possibly irrational,* coefficients β_s in the convex combination. Roughly speaking, this follows from the fact that each real number can be approximated with rational numbers up to an arbitrary error. However, the proofs are technically more involved.

Folk Theorems – Long-Run Average & SPE

Theorem 61
Let s^* be a pure strategy Nash equilibrium in G and let $v = (v_1, v_2)$ be a feasible vector of payoffs satisfying $v_i \geq u_i(s^*)$ for both $i \in \{1, 2\}$. Then there is a strategy profile $\tau = (\tau_1, \tau_2)$ in G_{irep} such that

- *τ is a SPE in G_{irep}^{avg}*
- *$u_i^{avg}(\tau) = v_i$ for $i \in \{1, 2\}$*

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- τ *is a SPE in* G_{irep}^{avg}
- $u_i^{avg}(\tau) = v_i$ *for* $i \in \{1, 2\}$

Proof: Consider a strategy profile $\tau = (\tau_1, \tau_2)$ in G_{irep} which gives the following behavior:

1. Unless one of the players deviates, the players play **cyclically** all profiles $s \in S$ so that each s is always played for γ_s rounds.
2. Whenever one of the players deviates, then, from that moment on, each player i plays s_i^*.

It is easy to see that $u_i^{avg}(\tau) = v_i$.

We verify that τ is SPE.

Folk Theorems – Long-Run Average & SPE

Fix a history h, we show that $\tau^h = (\tau_1^h, \tau_2^h)$ is a NE in G_{irep}^{avg}.

Folk Theorems – Long-Run Average & SPE

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- If h does not contain any deviation from the cyclic behavior 1., then τ^h continues according to 1., thus $u_i^{avg}(\tau^h) = v_i$.

Folk Theorems – Long-Run Average & SPE

Fix a history h, we show that $\tau^h = (\tau_1^h, \tau_2^h)$ is a NE in G_{irep}^{avg}.

- If h does not contain any deviation from the cyclic behavior 1., then τ^h continues according to 1., thus $u_i^{avg}(\tau^h) = v_i$.
- If h contains a deviation from 1., then

  $$w_{\tau^h} = s^* s^* \cdots$$

  and thus $u_i^{avg}(\tau^h) = u_i(s^*)$.

Folk Theorems – Long-Run Average & SPE

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- If h contains a deviation from 1., then

  $$w_{\tau^h} = s^* s^* \cdots$$

  and thus $u_i^{avg}(\tau^h) = u_i(s^*)$.
- Now if a player i deviates from τ_i^h to $\bar{\tau}_i^h$ in G_{irep}^{avg}, then

  $$w_{(\bar{\tau}_i^h, \tau_{-i}^h)} = \alpha(s_i^1, s'_{-i})(s_i^2, s^*_{-i})(s_i^3, s^*_{-i}) \cdots$$

  where α is a sequence of profiles following the cyclic behavior 1., $s_i^1, s_i^2, \ldots$ are strategies of S_i and s'_{-i} is a strat. of S_{-i}.

Folk Theorems – Long-Run Average & SPE

Fix a history h, we show that $\tau^h = (\tau_1^h, \tau_2^h)$ is a NE in G_{irep}^{avg}.

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- If h contains a deviation from 1., then

  $$w_{\tau^h} = s^* s^* \cdots$$

  and thus $u_i^{avg}(\tau^h) = u_i(s^*)$.
- Now if a player i deviates from τ_i^h to $\bar{\tau}_i^h$ in G_{irep}^{avg}, then

  $$w_{(\bar{\tau}_i^h, \tau_{-i}^h)} = \alpha(s_i^1, s'_{-i})(s_i^2, s^*_{-i})(s_i^3, s^*_{-i}) \cdots$$

  where α is a sequence of profiles following the cyclic behavior 1., $s_i^1, s_i^2, \ldots$ are strategies of S_i and s'_{-i} is a strat. of S_{-i}.

  However, then $u_i^{avg}(\bar{\tau}_i^h, \tau_{-i}^h) \le u_i(s^*) \le v_i$ since s^* is a Nash equilibrium and thus $u_i(s_i^k, s^*_{-i}) \le u_i(s^*)$ for all $k \ge 1$.

  Intuitively, player $-i$ punishes player i by playing s^*_{-i}. $\square$

Folk Theorem – Example

Consider the infinitely repeated game G_{irep} based on the following game G:

	m	f	r
M	$4,4$	$-1,5$	$3,0$
F	$5,-1$	$1,1$	$0,0$
R	$0,3$	$0,0$	$2,2$

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R	$0,3$	$0,0$	$2,2$

NE in $G : (F, f)$

Consider the grim trigger for (M, m) using (F, f), i.e., the profile (τ_1, τ_2) in G_{irep} where

- τ_1 : Plays M in a given stage if (M, m) was played in all previous stages, and plays F otherwise.
- τ_2 : Plays m in a given stage if (M, m) was played in all previous stages, and plays f otherwise.

Folk Theorem – Example

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NE in G : (F, f)

Consider the grim trigger for (M, m) using (F, f), i.e., the profile (τ_1, τ_2) in G_{irep} where

- τ_1 : Plays M in a given stage if (M, m) was played in all previous stages, and plays F otherwise.
- τ_2 : Plays m in a given stage if (M, m) was played in all previous stages, and plays f otherwise.

This is a SPE in G_{irep}^{avg}. Also, $u_i(\tau_1, \tau_2) = 4$ for $i \in \{1, 2\}$.

Are there other SPE?

Folk Theorem – Example

Consider the infinitely repeated game G_{irep} based on the following game G:

	m	f	r
M	$4,4$	$-1,5$	$3,0$
F	$5,-1$	$1,1$	$0,0$
R	$0,3$	$0,0$	$2,2$

NE in G : (F, f)

Consider the grim trigger for (M, m) using (F, f), i.e., the profile (τ_1, τ_2) in G_{irep} where

- τ_1 : Plays M in a given stage if (M, m) was played in all previous stages, and plays F otherwise.
- τ_2 : Plays m in a given stage if (M, m) was played in all previous stages, and plays f otherwise.

This is a SPE in G_{irep}^{avg}. Also, $u_i(\tau_1, \tau_2) = 4$ for $i \in \{1, 2\}$.

Are there other SPE? Yes, a grim trigger for (R, r) using (F, f). This is also a SPE in G_{irep}^{avg}.

Tacit Collusion

Consider the Cournot duopoly game model $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \kappa]$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

If the firms sign *a binding contract* to produce only $\theta/4$, their profit would be $\theta^2/8$ which is higher than the profit $\theta^2/9$ for playing the NE $(\theta/3, \theta/3)$.

However, such contracts are forbidden in many countries (including US).

Is it still possible that the firms will behave selfishly (i.e. only maximizing their profits) and still obtain such payoffs?

In other words, is there a SPE in the infinitely repeated game based on G which gives the payoffs $\theta^2/8$?

Tacit Collusion

Consider the Cournot duopoly game model $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Consider the grim trigger profile for $(\theta/4, \theta/4)$ using $(\theta/3, \theta/3)$:
Player i will

- produce $q_i = \theta/4$ whenever all profiles in the history are $(\theta/4, \theta/4)$,
- whenever one of the players deviates, produce $\theta/3$ from that moment on.

Tthis is a SPE in G_{irep}^{avg}. It results in $(\theta/4, \theta/4)(\theta/4, \theta/4)\cdots$ with the long-run average payoffs $\theta^2/8$.

Folk Theorems – Individually Rational Payoffs

Definition 62

$v = (v_1, v_2) \in \mathbb{R}^2$ is *individually rational* if for both $i \in \{1, 2\}$ holds

$$v_i \geq \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$$

That is, v_i is at least as large as the value that player i may secure by playing best responses to the most hostile behavior of player $-i$.

Folk Theorems – Individually Rational Payoffs

Definition 62
$v = (v_1, v_2) \in \mathbb{R}^2$ is *individually rational* if for both $i \in \{1, 2\}$ holds

$$v_i \geq \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$$

That is, v_i is at least as large as the value that player i may secure by playing best responses to the most hostile behavior of player $-i$.

Example:

	L	R
U	$-2, 2$	$1, -2$
M	$1, -2$	$-2, 2$
D	$0, 1$	$2, 3$

Here any (v_1, v_2) such that $v_1 \geq 1$ and $v_2 \geq 2$ is individually rational.

Folk Theorems – Long-Run Average & NE

Theorem 63

Let $v = (v_1, v_2)$ be a feasible and individually rational vector of payoffs. Then there is a strategy profile $\tau = (\tau_1, \tau_2)$ in G_{irep} such that

- *τ is a Nash equilibrium in G_{irep}^{avg}*
- *$u_i^{avg}(\tau) = v_i$ for $i \in \{1, 2\}$*

Folk Theorems – Long-Run Average & NE

Theorem 63
Let $v = (v_1, v_2)$ be a feasible and individually rational vector of payoffs. Then there is a strategy profile $\tau = (\tau_1, \tau_2)$ in G_{irep} such that

- τ *is a Nash equilibrium in* G_{irep}^{avg}
- $u_i^{avg}(\tau) = v_i$ *for* $i \in \{1, 2\}$

Proof: It suffices to use a slightly modified strategy profile $\tau = (\tau_1, \tau_2)$ in G_{irep} from Theorem 61:

- Unless one of the players deviates, the players play **cyclically** all profiles $s \in S$ so that each s is always played for γ_s rounds.
- Whenever a player i deviates, the opponent $-i$ plays a strategy $s_{-i}^{\min} \in \operatorname{argmin}_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$.

It is easy to see that $u_i^{avg}(\tau) = v_i$.

If a player i deviates, then his long-run average payoff cannot be higher than $\min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i}) \le v_i$, so τ is a NE. □

Folk Theorems – Long-Run Average & NE

Theorem 64
If a strategy profile $\tau = (\tau_1, \tau_2)$ *is a NE in* G^{avg}_{irep}*, then* $\left(u^{avg}_1(\tau), u^{avg}_2(\tau)\right)$ *is individually rational.*

Folk Theorems – Long-Run Average & NE

Theorem 64

If a strategy profile $\tau = (\tau_1, \tau_2)$ *is a NE in* G_{irep}^{avg}*, then* $\left(u_1^{avg}(\tau), u_2^{avg}(\tau)\right)$ *is individually rational.*

Proof: Suppose that $\left(u_1^{avg}(\tau), u_2^{avg}(\tau)\right)$ is not individually rational.

W.l.o.g. assume that $u_1^{avg}(\tau) < \min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2)$.

Theorem 64
If a strategy profile $\tau = (\tau_1, \tau_2)$ *is a NE in* G_{irep}^{avg}*, then* $\left(u_1^{avg}(\tau), u_2^{avg}(\tau)\right)$ *is individually rational.*

Proof: Suppose that $\left(u_1^{avg}(\tau), u_2^{avg}(\tau)\right)$ is not individually rational.
W.l.o.g. assume that $u_1^{avg}(\tau) < \min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2)$.

Now let us consider a new strategy $\bar{\tau}_1$ such that for every history h the pure strategy $\bar{\tau}_1(h)$ is a best response to $\tau_2(h)$.

Folk Theorems – Long-Run Average & NE

Theorem 64
If a strategy profile $\tau = (\tau_1, \tau_2)$ is a NE in G^{avg}_{irep}, then $\left(u^{avg}_1(\tau), u^{avg}_2(\tau)\right)$ is individually rational.

Proof: Suppose that $\left(u^{avg}_1(\tau), u^{avg}_2(\tau)\right)$ is not individually rational. W.l.o.g. assume that $u^{avg}_1(\tau) < \min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2)$.

Now let us consider a new strategy $\bar{\tau}_1$ such that for every history h the pure strategy $\bar{\tau}_1(h)$ is a best response to $\tau_2(h)$.

But then, for every history h, we have

$$u_1(\bar{\tau}_1(h), \tau_2(h)) \geq \min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2) > u^{avg}_1(\tau)$$

So clearly $u^{avg}_1(\bar{\tau}_1, \tau_2) > u^{avg}_1(\tau)$ which contradicts the fact that (τ_1, τ_2) is a NE. $\square$

Note that if irrational convex combinations are allowed in the definition of feasibility, then vectors of payoffs for Nash equilibria in G^{avg}_{irep} are exactly feasible and individually rational vectors of payoffs. Indeed, the coefficients β_s in the definition of feasibility are exactly frequencies with which the individual profiles of S are played in the NE.

Folk Theorems – Summary

- We have proved that "any reasonable" (i.e. feasible and individually rational) vector of payoffs can be justified as payoffs for a Nash equilibrium in G_{irep}^{avg} (where the future has "an infinite weight").
- Concerning SPE, we have proved that any feasible vector of payoffs dominating a Nash equilibrium in G can be justified as payoffs for SPE in G_{irep}^{avg}.

  This result can be generalized to arbitrary feasible and *strictly* individually rational payoffs by means of a more demanding construction.
- Folks theorems can also be formulated for (a more common) discounted payoff defined by $u_i^{\delta}(\tau) = (1-\delta)\sum_{t=0}^{\infty} \delta^t \cdot u_i(s^{t+1})$. The results (more or less) coincide with the long-run average versions as δ goes to 1.

For a very detailed discussion of Folk Theorems see "A Course in Game Theory" by M. J. Osborne and A. Rubinstein.

Summary of Extensive-Form Games

We have considered extensive-form games (i.e., games on trees)

- with perfect information
- with imperfect information

We have considered pure strategies, mixed strategies and behavioral strategies (Kuhn's theorem).

We have considered Nash equilibria (NE) and subgame perfect equilibria (SPE) in pure strategies.

Summary of Extensive-Form Games (Cont.)

For perfect information we have shown that

- there always exists a pure strategy SPE
- SPE can be computed using backward induction in polynomial time

For imperfect information the following holds:

- The backward induction can be used to propagate values through "perfect information nodes", but "imperfect information parts" have to be solved by different means
- Solving imperfect information games is at least as hard as solving games in strategic-form; however, even in the zero-sum case, most decision problems are NP-hard.

Summary of Extensive-Form Games (Cont.)

Finally, we discussed repeated games. We considered both, finitely as well as infinitely repeated games.

For finitely repeated games we considered the average payoff and discussed existence of pure strategy NE and SPE with respect to existence of NE in the original strategic-form game.

For infinitely repeated games we considered both

- discounted payoff: We have formulated and applied a simple folk theorem: "grim trigger" strategy profiles can be used to implement any vector of payoffs strictly dominating payoffs for a Nash equilibrium in the original strategic-form game.
- long-run average payoff: We have proved that all feasible and individually rational vectors of payoffs can be achieved by Nash equilibria (a variant of grim trigger).